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EDITED BY

EARLE R. HEDRICK
DAVID R. CURTISS WILLIAM R. LONGLEY

WITH THE ASSISTANCE OF

HAROLD T. DAVIS HARRY W. KUHN
THEOPHIL H. HILDEBRANDT DERRICK N. LEHMER
JOHN R. KLINE HENRY L. RIETZ

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ERNEST BROWN SKINNER—IN MEMORIAM

Ernest Brown Skinner was born December 12, 1863, at Redfield, Ohio, and died in Madison, Wisconsin, April 3, 1935, after a brief illness of five days. He was the eldest son of a sturdy Ohio farmer. Two of his four brothers are engaged very successfully in educational or scientific work—B. O. Skinner, State Director of Education in Ohio, and C. E. Skinner, Assistant Director of Engineering in the Westinghouse Electric and Manufacturing Company.

Professor Skinner graduated at Ohio University, Athens, Ohio, in 1888, and shortly afterwards married a college-mate, Adda C. Coe. Together they shared a most happy home life for forty-seven years. Three children, two daughters and a son, survive him.

For three years after graduation Professor Skinner taught at Amity College in Iowa. He was then in 1891 awarded a scholarship at Clark University, Worcester, Massachusetts, and began his mathematical specialization, largely under Story and Bolza. After a year his study there was broken off because of the offer of an instructorship at the University of Wisconsin. The acceptance of this offer was decisive in his career since it connected him with the institution to which he gave loyal and enthusiastic devotion for forty-three years. His service at Wisconsin was interrupted only by leave of absence for the academic year 1899–1900. By combining this with the preceding and following summer sessions at the University of Chicago, he completed successfully his work for the Ph.D. degree. His thesis bore the title *On ternary monomial substitution-groups of finite order with determinant* ± 1, and was published in the American Journal of Mathematics, vol. 25 (1902). In succeeding years he remembered with great gratitude and affection those stimulating teachers Maschke, Bolza, and Moore. The lively interest in the theory of groups which these professors inspired in him made this subject henceforth his favorite field of study. The article upon *Groups* in the last edition (1929) of the *Encyclopaedia Britannica* is from his pen.

In the University of Wisconsin Professor Skinner was successively promoted to an assistant professorship in 1895, to an associate professorship in 1910, to a full professorship in 1920. About 1909 he accepted the onerous task of coordinating and supervising the work in freshman mathematics in the College of Letters and Science, given in some twenty-five divisions under instructors of varied training and ability. For such supervision he was extraordinarily adapted by his sympathy and approachability, his fairness and judgment. The young instructors and graduate assistants working under him relied on his guidance and strength and steadily developed in efficiency. Instructor and student alike knew that there was a strong man at the helm to whom they could always go for advice and encouragement.

Largely because of this supervision of freshman work, Professor Skinner wrote several college textbooks: a *College Algebra* (1917) and an *Introduction to Trigonometry and Analytic Geometry* (1932). But foremost and of outstanding significance for college instruction must be placed his book, *The Mathematical Theory of Investment* issued first in 1913 and later, somewhat

revised, in 1924. This was for America a pioneer in the field called by Germans *Political Arithmetic* and was designed particularly for University students in commercial courses. It admirably met a pressing need and was given the compliment of becoming the model for other texts on the same subject.

A natural outcome of this work in technical mathematics was Professor Skinner's intense interest in the pension and annuity systems for teachers in his university and state. Upon the establishment of the University Retirement Board in 1921 he became its chairman, and he never ceased thereafter to watch with great solicitude over the pension and annuity funds for teachers of the state. Among many other important committee assignments in the university may be singled out his membership in the dean's Executive Committee in the College of Letters and Science. His judgment, human contact, and knowledge of university conditions were so indispensable to this committee that he was retained as a member from the time of first appointment in 1902 until his retirement as professor emeritus in 1934, being the sole member so retained. In the decade when the schools of the state were visited by college professors for certification under the entrance requirements of the university, he was repeatedly employed as an inspector, in which position he became by his tact and judgment more an ambassador of good will than an inspector. His tours over the state and his wide and friendly contact with a great number of freshman students made him in the State of Wisconsin the most widely known and best beloved member of the mathematical staff of the university. When the sub-committee on State Inspection and Supervision of Instruction was formed by the American Commissioners of the International Commission on the Teaching of Mathematics, Professor Skinner was appropriately selected to be the chairman of this sub-committee and wrote its report.

In 1919 Skinner was elected by the City of Madison a member of its Board of Education and fortunately served as its president during nine formative and critical years in the school system of a rapidly growing city. In this position he singularly commanded the confidence of both workingmen and scholars.

Among other connections I may mention his five years as Secretary and Editor of the Wisconsin Academy of Sciences, Arts, and Letters, his twenty-eight-year directorship in the McCormick Theological Seminary, and his terms in the General Assembly of the Presbyterian Church of the U.S.A. with membership in its Judicial Commission for the years 1926–1929. He was a member of the American Mathematical Society and also of the Mathematical Association, being a regular attendant at the Chicago meetings, where he won the high regard and affection of his fellow mathematicians. Ohio University conferred on him the honorary degree of LL.D. in 1932.

The counsel and advice of Professor Skinner were much sought and always highly valued by his colleagues. Broad in sympathy and intellectual outlook, winning in personality, absolutely unselfish and dependable both as teacher and colleague, he gave his all modestly and lavishly. Most fittingly it can be said of Ernest Brown Skinner with his high ideals and robust character that he was "the type of man who makes the backbone of a university."

E. B. VAN VLECK

WIENER ON THE FOURIER INTEGRAL

The Fourier Integral and Certain of its Applications. By Norbert Wiener. Cambridge University Press, 1933. xi+201 pp.

An analyst of the older generation would probably understand the term Fourier integral to refer to the formula

$$f(x) = \frac{1}{\pi}\int_0^\infty du \int_{-\infty}^\infty f(t)\cos u(t-x)dt,$$

which is valid for a very restricted class of functions. He would not find this formula in Wiener's treatise, which is really devoted to the theory of Fourier transforms, and the corresponding reciprocity relations. The Fourier transform of $f(x)$ is

$$g(x) = (2\pi)^{-1/2}\int_{-\infty}^\infty f(t)e^{-ixt}dt \tag{1}$$

with suitable interpretation of the integral which ordinarily does not exist in the sense of Lebesgue. The reciprocity theorem states that conversely $f(x)$ is the Fourier transform of $g(-x)$ with, conceivably, a different interpretation of the integral.

Wiener is concerned almost exclusively with three classes of measurable functions $f(x)$, namely, L_1, L_2, and S. Here the first two symbols have their customary meaning, S is the class of functions of uniformly bounded mean square modulus. In the first case (1) is an ordinary Lebesgue integral, $g(x)$ is continuous and vanishes at infinity. In the second case the integrals are limits in the mean of order two, and the transform belongs to L_2 [Plancherel's theorem]. The transformation is unitary so that $f(x)$ and $g(x)$ have the same norm in L_2. In the third case, the integral of the formal transform serves the same needs.

The book is grouped into an Introduction and four chapters. Of these Chapter 1 uses ideas from a paper by the author on Plancherel's theorem (Journal of Mathematics and Physics, M.I.T., vol. 7 (1928)). Chapters 2 and 3 are an elaboration of portions of the author's Bôcher Prize memoir (Annals of Mathematics, (2), vol. 33 (1932)). The last chapter is based on another paper by the author (Acta Mathematica, vol. 55 (1930)). The Introduction [45 pp.] gives a rapid outline of the main theorems on Lebesgue and Riemann-Stieltjes integrals, orthogonal series, and the Riesz-Fischer theorem. We note in passing that Theorem X_{21} is false, but as the author neither proves nor uses it, no harm is done.

Chapter 1 [26 pp.] is devoted to Plancherel's theorem mentioned above. The proof given by the author (one of his own; he has given several others) is ultimately based upon the fact that the function $\exp(-x^2/2)$ is its own Fourier transform and on related properties of Hermitian polynomials. If $f(x)\epsilon L_2$, and

$$f(x) \sim \sum_{n=0}^\infty f_n\psi_n(x), \tag{2}$$

where $\{\psi_n(x)\}$ is the orthonormal set of Hermitian functions, then

$$g(x) \sim \sum_{n=0}^{\infty} (-i)^n f_n \psi_n(x), \tag{3}$$

the series being Abel summable as well as convergent in the mean. Formulas (2) and (3) show the unitary character of the Fourier transformation at a glance.

Chapter 2 [32 pp.] deals with the *General Tauberian theorem*. This fundamental discovery, which the author made in 1926–27, is concerned with the limiting properties of linear transformations of the type

$$K[f] = \int_{-\infty}^{\infty} K(x-t)f(t)dt, \tag{4}$$

where $K[1]=1$. Suppose that for a bounded function $f(x)$

$$\lim_{x\to\infty} K[f] = A. \tag{5}$$

Can any conclusion be drawn about the existence of such a limit if the kernel $K(u)$ is replaced by another kernel $K_1(u)$? Wiener's answer is that (5) will hold with K replaced by K_1 provided both kernels belong to L_1, and the Fourier transform of $K(u)$ vanishes for no real value of x. Moreover, if the Fourier transform has a real zero, there exists at least one kernel $K_1 \epsilon L_1$, $K_1[1] = 1$, for which (5) is not true. The theorem appears to lie quite deep; the proof requires a very elaborate analysis, and led the author to a number of interesting theorems on absolutely convergent Fourier series and closure in the space L_1.

Wiener's basic theorem seems rather far removed from Tauber's original theorem of 1897 or its later refinements, but this is actually the first application in Chapter 3, *Special Tauberian theorems* [46 pp.]. This is followed by various important applications to analytical number theory. The author gives two distinct reductions of the prime-number theorem, in the form due to Hadamard and de la Vallée Poussin, to Tauberian form. This includes a proof of the Ikehara-Landau theorem. He does not expect much from Tauberian methods, however, when it comes to the more refined theorems on the distribution of the primes, in particular those theorems which involve the Riemann hypothesis.

Chapter 4, *Generalized harmonic analysis* [50 pp.], deals mainly with the functions of class S. This discussion really starts in the last paragraph of Chapter 3 where the mean square modulus and an integrated Fourier transform are brought into play. The principal role is played by the function $s(u+\epsilon)-s(u-\epsilon)$, which is the ordinary Fourier transform of $2f(x)\sin \epsilon x/x$ in L_2. The author defines the "spectrum" of $f(x)$ to be the function

$$\sigma(u) = \text{const.} + \text{l.i.m.}_{\epsilon\to 0} \frac{1}{2\epsilon(2\pi)^{1/2}} \int_0^u |s(t+\epsilon) - s(t-\epsilon)|^2 dt. \tag{6}$$

This expression determines the energy distribution of the oscillation $v=f(t)$, v being the displacement and t the time. If $f(x)$ be subjected to a transformation of type (4), with suitable restrictions on the kernel, a function of class S will again result whose spectrum is easily computed from that of $f(x)$. The local deformation of the spectrum is proportional to the square of the modulus

of the Fourier transform of the kernel. These ideas are finally applied to a derivation of the basic theorems on uniformly almost periodic functions.

This rapid survey will give some notion of the bewildering wealth of new ideas in Professor Wiener's book. The reader who approaches this book for the first time should realize in advance that he is not holding in his hands a compendium of the theory of Fourier transforms, complete up to July 1932, but a report from the workshop of one of the investigators who has perhaps done more than anybody else to establish the Fourier integral firmly as an indispensable tool in modern analysis. The book under review contains the main gems which he has produced with his tools up to the date mentioned. We all know that the production is still going on, and the recent Colloquium lectures of the author and the late R.E.A.C. Paley show no signs of any approaching dullness of the tools nor of the wits behind them. It lies in the nature of the subject that the book is not easy reading, but the Introduction and Chapter 1 are well suited for the needs of a student who wants to learn the elements of the theory.

The lamented Paley used to amuse his friends by stating two criteria for estimating the value of a mathematical contribution. According to the first, one computed the essential theorem density, that is, the number of significant theorems divided by the number of pages. In the second criterion, the essential quotation density, the number of theorems was replaced by the number of references to Paley's work! If one of those who also writes on Fourier integrals should apply these tests, mutatis mutandis, to Wiener's book, the score would come out very high according to the first one, but fairly low according to the second.* The book is singularly autarchic. This is in some sense a weakness; the reader would have gained here and there by a broader outlook. But the reader has to keep in mind the author's stated purpose. He wanted to present his own investigations in book form with the prefatory material necessary to follow the argument, and he has succeeded admirably. It is not a text book, nor a monograph on all the central features of the theory of Fourier integrals. But a person who wants to learn what Fourier integrals are, what they are good for, and who aspires to mastery of the new technique involved in handling them, will do well to sit down and read this book patiently.

EINAR HILLE

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* If the reviewer had got his dues on p. 70, he would have been able to raise the second score from 0 to 1/212.

QUINE ON LOGISTIC

A System of Logistic. By Willard Van Orman Quine. Harvard University Press, 1934. x+204 pp.

In this book is presented a system of symbolic logic based on that of Whitehead and Russell's *Principia Mathematica*, but involving a number of fundamental changes. The most important of these changes are: (1) the representation of functions of two or more variables as functions of one variable through the introduction, as an undefined term, of the operation of ordination, that is, the operation of combining two elements a and b into the ordered pair a,b; (2) the use of this same notion of ordination to replace the notion of predication, the proposition ϕa, obtained by predicating the propositional function ϕ of the argument a, being identified with the ordered pair ϕ,a; (3) the introduction in connection with the operation of abstraction, $\hat{}$, of a rule of inference, the rule of concretion, which takes the place of that tacit rule of *Principia* which, to speak somewhat inexactly, allows the substitution for ϕx, in any proved expression in which ϕ is a free variable, of any appropriate expression containing x; (4) a liberalization of the theory of types, by which the axiom of reducibility is rendered unnecessary; (5) the use of the notion of classial referent, introduced by an actual nominal definition, to replace almost entirely the clumsy descriptions introduced in *Principia* as incomplete symbols; (6) the introduction, under the name of congeneration, of the relation of implication between propositional functions, as an undefined term, out of which both the relation of implication between propositions and the universal and existential quantifiers are obtained by definition.

Quine's propositional functions have the property that equivalence implies equality, and for this reason he speaks of them as classes rather than as propositional functions. Nevertheless he uses them for the purposes for which propositional functions are used in *Principia* and in other systems, and hence, for the sake of comparison, we continue to call them propositional functions.

In regard to Quine's use of ordination, it is, of course, clear, as he points out, that the introduction as primitive ideas of an infinite number of different notions of predication, one for functions of one variable, another for functions of two variables, another for functions of three variables, and so on, is awkward and that it is therefore desirable to find some device by which functions of two or more variables can be regarded as special cases of functions of one variable. It is not so clear, however, that the introduction of the ordered pair as an undefined term is the best method of doing this. From some points of view the more natural and more elegant method is that of Schönfinkel,* under which a function of $n+1$ variables is regarded as a function of one variable whose values are functions of n variables. For example, instead of what is ordinarily written $\phi(a, b)$, Schönfinkel writes $(\phi a)b$, where ϕa is regarded as a function which, when taken of the argument b, yields the proposition $(\phi a)b$, and ϕ is regarded as a function which, when taken of the argument a, yields the func-

* Mathematische Annalen, vol. 92 (1924), pp. 305–316.

tion ϕa. Of course, if the Schönfinkel device be adopted, it is necessary to introduce the notion of predication for functions of one variable as a primitive idea, but the number of primitive ideas is not thereby increased, because the ordered pair a,b can then be defined, in terms of predication and abstraction, as $\hat{\phi}((\phi a)b)$, that is, as the class of relations which hold between a and b.

On the other hand, Quine's identification of the notion of predication with that of ordination not only raises the difficult philosophical problem of justifying the assumption that, if x is of one type higher than y, then any assertion about x and y can be construed as an assertion about the proposition x,y, but also introduces unnecessary formal complications, for example, in the rule of concretion. The use of predication as a primitive idea has the advantage that it neither accepts nor denies Quine's special philosophy concerning the nature of predication.

The use of the device of Schönfinkel just referred to is not incompatible with the theory of types, but it does require modification of the usage of *Principia* by which propositional functions are regarded as entities of an entirely different sort from other functions, modification at least to the extent of allowing that functions of one variable whose values are propositions, and functions of one variable whose values are propositional functions, are concepts sufficiently similar so that one notion of application, or predication, and one method of symbolizing this notion, are sufficient for both.

As a matter of fact, it is the contention of the present reviewer that the distinction between propositional functions and functions of other sorts is no more fundamental than, say, the distinction between functions of a real variable and functions of a complex variable, and that the one notion of application, or predication, should suffice for all functions of one variable. Quine, however, following *Principia*, has two notations for the application of a function, one for propositional functions and one for descriptive functions, and in the same way two notations for the operation of abstraction. Thus if M is an expression which contains x as a free variable and which takes on propositions as values when x takes on particular values, he uses $\hat{x}\mathrm{M}$ to denote the corresponding propositional function, and $\hat{x}\mathrm{M},a$ to denote the result of application of this propositional function to the argument a. But if M takes on classes as values when x takes on particular values, then he uses

$$\hat{w}(\exists,\hat{\alpha}(\exists,x(w = \alpha,x \cdot \alpha = \mathrm{M})))$$

to denote the corresponding function (his usage amounts to that), and $\hat{w}(\exists,\hat{\alpha}(\exists,\hat{x}(w=\alpha,x\cdot\alpha=\mathrm{M})))'a$ to denote the result of application of this function to the argument a.

The rule of inference of *Principia* referred to under (3) in the first paragraph above is (like the simple rule of substitution) entirely suppressed by the authors of that work, who use it repeatedly but make no mention of it. Hilbert and Ackermann* make this rule explicit, but their statement of it is inadequate. Quine's revised statement of the rule, on page 187 of his book, is perhaps adequate as applied to the system of *Principia*, but his statement of

* *Grundzüge der Theoretischen Logik*, p. 53.

what the analogous rule would have to be for his own system is again inadequate, as can be seen, for example, by considering the proposition

$$\alpha,\hat{v}(\alpha,\iota' v) \cdot \supset : \cdot \backsim \cdot \alpha,\wedge : \supset \cdot \exists,\hat{w}(\alpha,\iota' w)$$

and raising the question of substituting for the free variable α on the basis that α,x shall mean $x,\hat{y}(x,[y])$. The following is proposed as a correct statement of what this rule of inference should be for use in Quine's system (in order to make possible in Quine's system the equivalent of what Quine's (26) on page 187 makes possible in the system of *Principia*).

Let A be a significant expression, and let α be a variable whose occurrences as a free variable in A are occurrences as the first symbol of parts of A of the form α,U. We may list these parts as $\alpha,U_1, \cdots, \alpha,U_n$, where $U_1, \cdots, U_n$ are significant, and where the listing is in such an order that, if U_j contains α,U_i, then $i<j$. Let M be a significant expression in which x occurs as a free variable, and let M_1 stand for the result of substituting (in the sense of Quine, page 42) U_1 for x in M. Let A_1 be the expression obtained from A by substituting M_1 for the part α,U_1 and let $\alpha,U_{12}, \cdots, \alpha,U_{1n}$ be the parts of A_1 into which the parts $\alpha,U_2, \cdots, \alpha,U_n$ of A are transformed by the substitution. Let M_2 stand for the result of substituting U_{12} for x in M. Let A_2 be the expression obtained from A_1 by substituting M_2 for the part α,U_{12}, and let $\alpha,U_{22}, \cdots, \alpha,U_{2n}$ be the parts of A_2 into which the parts $\alpha,U_{13}, \cdots, \alpha,U_{1n}$ of A_1 are transformed by the substitution, and so on, until M_n and A_n are defined. If A is a proved expression, the rule which we are stating allows us to infer A_n, provided that A_n is a propositional expression (as defined below).*

The superior simplicity of the rule of concretion, and the advantage of avoiding the foregoing complicated rule by introducing the simpler one (as Quine does), is obvious. In fact, the effect is to analyze a complicated inference into a series of simpler inferences by substitution and concretion, as can be illustrated in connection with the example just mentioned, by substituting $\hat{x}(x,\hat{y}(x,[y]))$ for α in the proposition in question and then making a number of successive applications of the rule of concretion (four or five according to the order in which they are made). Unfortunately, as already remarked, the essential simplicity of the rule of concretion is partially obscured by the peculiar use of ordination as a substitute for predication.

Nowhere in Quine's book is there a definition of the word *significant*, which is used in his statement of the rule of substitution, but from scattered remarks about types and by observation of how the rule of substitution is actually used, it is possible to surmise what probably is meant by the word. Since this is a matter of some importance, especially in view of the fact that it is only through this word (or the related term *propositional expression*) that the theory of types enters the formal system at all, an explicit definition of *significant* is attempted here.

The four metamathematical (or "prosystematic") terms, *significant, classial expression, propositional expression, type,* must be defined simultaneously by induction, as follows. A variable standing alone is significant and may be as-

* With appropriate modifications to adapt it to the notation of Quine, the statement of this rule is taken from a set of notes by S. C. Kleene on lectures of Kurt Gödel, which the reviewer has before him.

signed any type out of the scheme of types explained in Quine's second chapter; if assigned a type of the form $a!$, the variable is a classial expression, and if assigned a type of the form $a! \uparrow a$, it is a propositional expression. If M and N are significant and are assigned the types m and n, respectively, and if it is true of every variable x which occurs in both M and N that the same type was assigned to x in assigning the type m to M that was assigned to x in assigning the type n to N, then (M, N) is significant and must be assigned the type $m \uparrow n$; moreover, if m is $n!$, then (M,N) is a propositional expression. If M is assigned the type m and is a classial expression, then [M] is significant, is a classial expression, and must be assigned the type $m!$. If M is assigned the type m and is a propositional expression, and x is any variable, then $\hat{x}$M is significant, is a classial expression, *and must be assigned the type* $r!$ *where* r *is the type that was assigned to* x *in assigning the type* m *to* M, *or, if* x *does not occur in* M, *where* r *is any type whatever*. When no particular assignment of types to the parts of an expression is in question, the expression shall be called significant (a classial expression, a propositional expression) if types can be assigned to its parts so as to make it significant (a classial expression, a propositional expression).

Quine further requires that an expression set down as a postulate or theorem shall not be considered significant unless it is a propositional expression. But this seems to be an unnecessary complication of terminology, which could be avoided by no greater change than replacing the word "significant" by "a propositional expression" in the statement of the rule of substitution.

The italicized clause in the foregoing definition marks a sharp divergence of Quine's theory of types from that of *Principia Mathematica;* for if the analogy with the theory of types of *Principia* were preserved, $\hat{x}$M could not be of lower type than M.* It is true, of course, in *Principia*, that if α is a class then a proposition of the form $x \epsilon \alpha$ must be of type just one higher than the type of x, but it is to be remembered that this situation is brought about only with the aid of the axiom of reducibility, and that, in any case, the classes of *Principia* are incomplete symbols defined only contextually. Since Quine's $\hat{x}$M is not an incomplete symbol, a truer comparison of the two systems appears to be obtained if we compare expressions in Quine of the form $\hat{x}$M with the propositional functions of *Principia* rather than with the classes of *Principia*. And from this point of view it is seen that, without claiming to do so, Quine has really made an important modification in the theory of types, in a direction which seems to have been first suggested by F. P. Ramsey.†

This modification in the theory of types renders the axiom of reducibility unnecessary in the system of Quine. In particular, the difficulty in regard to the least upper bound of a bounded set of real numbers‡ disappears. For let real numbers be segments of rational numbers, and let λ be a bounded set of real

* See *Principia Mathematica*, 2d ed., vol. 1, p. 48 et seq., and introduction to the second edition, p. xxxix.

† Proceedings of the London Mathematical Society, (2), vol. 25 (1926), p. 362 et seq.

‡ See *Principia Mathematica*, introduction to the second edition, pp. xliv–xlv.

numbers. Then $\epsilon'\lambda$ (replacing the $s'\lambda$ of *Principia*) is the least upper bound of λ, and is of the same type as the real numbers in the set λ.

Quine's statements of his rules of inference require a number of corrections. In his definition, in connection with the rule of substitution, of what he means by the process of substitution, it should be provided that when the bound variable is rewritten in E′ it should be made alphabetically distinct, not only from all variables in E, but also from all variables previously occurring in E′. In the rule of subsumption the proviso should be added that $[\alpha]$, $\hat{x}(\text{- - -})$ be a propositional expression; otherwise, since $x,\alpha\cdot\supset\cdot x,\alpha$ is provable, the rule could be used to infer $[\alpha],x(x,\alpha\cdot\supset\cdot x,\alpha)$. In the rule of concretion it should be stipulated that - - - and $\cdots$ be significant; otherwise the rule could be applied to $\hat{z}(\hat{x}(x,y),z),\hat{t}(\exists,\hat{u}(u,t))$, taking - - - to be $(x,y),z$, and $\cdots$ to be $\hat{t}(\exists,\hat{u}(u,t))$.

Apparently a rule of inference allowing an alphabetical change of a bound variable in any theorem or postulate should be added to Quine's four rules, since such a rule of inference is used in the proof of 3.8 on page 83. Of course, it may be that such a rule of inference is unnecessary on the ground that the effect of it can be obtained by some succession of applications of the four rules of inference and the postulates. But if so, this should be explained.

The contention on page 51 that the use of z,x,y to replace $z,(x,y)$ is not to be construed as a definition, or abbreviation, is definitely untenable. For if z,x,y is to be regarded otherwise than as an abbreviated notation for $z,(x,y)$, the system of Quine is open to the same charge as that which he brings against the system of *Principia*, namely, that of being incompletely formalized and leaving lacunae to be bridged by the common sense of the reader. If z,x,y is not an abbreviation, and if the rule of substitution is to be taken literally, then we may substitute z,x for t in the proposition $t,y=u,\hat{p}(\sim p),(\sim\cdot v=v):\supset y$ and so obtain $z,x,y=u,\hat{p}(\sim p),(\sim\cdot v=v):\supset y$. In the opinion of the reviewer the remedy for this situation is to introduce the notation for an ordered pair as (x,y) instead of x,y, then to use x,y and z,x,y as abbreviations for (x,y) and $(z,(x,y))$, respectively, whenever no ambiguity is thereby created, of course with the understanding that the rules of inference are applicable only to the unabbreviated form of an expression. In this way the parentheses in (x, y) would become as much a part of the formal system as the brackets in $[\alpha]$, and the notion of parentheses as an extra-formal convention would disappear.

The classial referent of x with respect to the relation α, denoted by $\alpha'x$, is defined in such a way that its intuitive meaning is, "the class of all members of classes bearing the relation α to x." Thus, if there is one and only one class bearing the relation α to x, then $\alpha'x$ denotes that class. Consequently the classial referent can be used as a description in any case where the thing described is a class, and it happens in the system of Quine that nearly everything worth describing is a class. The superiority of a nominal definition over the use of descriptions as incomplete symbols (as in *Principia*) requires no elaboration.

A considerable economy in the number of primitive ideas is effected by introducing the notion of congeneration, denoted by [ ], as a primitive idea. This notion is explained by Quine on the basis that $[\alpha]$ means the class of classes containing α. But it may also be thought of as implication between propositional functions, because, if α and β are propositional functions (classes), then

$[\alpha],\beta$ is the proposition, "α implies β," expressed in *Principia* as $\alpha x \supset_x \beta x$. It is perhaps worth while to observe that $[\ \ ]$ is a propositional function of two variables, not in the sense of Quine, but in the sense of Schönfinkel, since, if α is a propositional function of one variable, $[\alpha]$ is a propositional function of one variable.

There is no slur on the invaluable pioneer work of Whitehead and Russell when it is said that their system is unsatisfactory from the viewpoints of formal definiteness and of mathematical elegance. The work of Quine is in both respects an important improvement over the system of *Principia*, and, although open to criticism in certain directions, is probably not too highly praised by Whitehead when he calls it, "A landmark in the history of the subject".

ALONZO CHURCH

AMERICAN MATHEMATICS BEFORE 1900

A History of Mathematics in America before 1900. By David Eugene Smith and Jekuthiel Ginsburg. (The Carus Monographs, No. 5.) Mathematical Association of America, 1934. x+209 pp.

The Committee on the Carus Monographs had a happy inspiration when it was led to induce Professor Smith to prepare this history. He was in every way qualified for the task—through his unique knowledge of the subject, through his attractive literary style, and through the excellence of his judgment in dealing with a great mass of material and in presenting its essence in well-balanced and compact form. All of these qualities are very much in evidence in the little volume under review. Only one who has had considerable experience in such matters can truly appreciate the great amount of research which went into the preparation of the manuscript. In this research Professor Smith had the valuable assistance of Professor Ginsburg of Yeshiva College, the editor-in-chief of Scripta Mathematica.

For the purposes of the history "America" was roughly considered as the territory north of the Caribbean Sea and the Rio Grande River. In 1938 fifty years of activity of the American Mathematical Society will be celebrated, and a number of scholars will doubtless cooperate in presenting a historical picture of each of the fields of American mathematics during that period. Such a survey, and the complementary work under review, will thus give an up-to-date panorama of outstanding mathematical activities of the past. The importance of these activities after 1875 for the extraordinary development in the twentieth century will be assessed, and Professor Smith's delineation of milestones of earlier progress will be recalled.

In the sixteenth and seventeenth centuries the mathematical needs of the early American settlers were few, and even at Harvard and William and Mary Colleges, nothing noteworthy was done. Astronomical observations were made to a certain extent, and almanacs prepared; astrologers were by no means unknown. "The century that saw the work of Galileo, Kepler, . . . ,* Napier,

* The name "Gilbert" occurred here in the original sentence (p. 13). The reviewer is unequal to guessing to whom it was intended to refer.

Fermat, Descartes, Pascal, Huygens, Newton and Leibniz, in countries from which the settlers had come, saw among the intelligentsia no apparent appreciation of the discoveries of scholars of this class."

The eighteenth century was the first in America when any special interest in mathematics was shown, and to this our authors devote 50 pages. The contributions from colleges, private instruction, textbooks, astronomy, navigation, and geodesy, learned societies and scientific periodicals, are all considered. Among outstanding individuals whose mathematical work, or influence on mathematics, is discussed, are Isaac Greenwood, John Winthrop, David Rittenhouse, Benjamin Franklin, and Thomas Jefferson.

The third chapter (pp. 65–101) gives us a general survey of the nineteenth century, including a further account of the mathematics of colleges and universities, and of scientific societies and periodicals. In the first complete paragraph on page 87 there is a call for revision throughout. It is not recognized that the periodicals Analyst or Mathematical Museum (5 nos., 150 pp., Philadelphia, 1808–11), and The Analyst (1 no., New York, 1814), were both edited by Robert Adrain. In this connection a change is also necessary in the first line on page 92. On page 88 the title of Marrat's The Scientific Journal is given incorrectly [see Mathematical Gazette, 1929, p. 393]. To introduce Gill's Mathematical Miscellany with "Among the problem-solving periodicals," is surely misleading. The powerful French influence in the early part of the century is not forgotten and prominent names during 1800–1875 are considered. Among these are Robert Adrain, Nathaniel Bowditch, Charles Gill,* and Ferdinand Rudolph Hassler. It is not clear why Benjamin Peirce is not listed here rather than later.

The fourth and final chapter (pp. 102–200) is devoted to the period 1875–1900. This began with the founding of the Johns Hopkins University, the appointment of its extraordinarily inspiring first professor of mathematics James Joseph Sylvester, and the founding of the American Journal of Mathematics. The New York (later American) Mathematical Society was founded in 1888 and soon the publication of its Bulletin was started. The great European (especially German) influence on American mathematics is indicated by the listing (without any pretense to completeness) of American mathematicians who got doctor's degrees from European universities before 1900. This list might have been increased by at least these four other names: T. H. Gronwall (Upsala) 1898; T. E. Hart (Heidelberg) 1866; H. D. Thompson (Göttingen) 1892; M. F. Winston (Göttingen) 1896. The name "Foche, Anne B. (Göttingen)," on the list, would not be recognized by those unaware that Anne Bosworth, who got the degree, later became the wife of Dean Foche.

Other influences in the development of mathematics of the period were the founding of such periodicals as the The Analyst, Annals of Mathematics, American Mathematical Monthly, W. E. Story's Mathematical Review, and Martin's Mathematical Magazine and Mathematical Visitor. It is stated that of the Review "only three numbers appeared"; more accurately this should read "only two numbers appeared, and some pages of a third number were

* References to Gill may be augmented by one to S. Neumark, *Note on the life of Charles Gill*, Scripta Mathematica, vol. 2, pp. 139–142.

printed, but never distributed." The impression is given that only Volume 1, containing 12 numbers, "1882–1884," of the Mathematical Magazine was ever published; the facts are that Volume 1 was printed during the years 1884–1887, and Volume 2 in 12 numbers, No. 12 in 3 parts, during the years 1890–1913. American contributions to British, but not to German, periodicals are noted.

"Prominent names and special interests" of the period makes fascinating reading. Of Sylvester, Cayley, Hill, Newcomb, Gibbs, W. W. Johnson, E. H. Moore, Maschke, Cole, W. I. Stringham, and J. S. Hagen, for example, there are brief biographies. An obvious slip was made in referring to Sylvester* as "second senior wrangler" in 1837. The volume number of the publication containing a biographical memoir of Gibbs is lacking. Stringham's excellent book, *Uniplanar Algebra: being Part I of a Propædeutic to Higher Mathematical Analysis* (San Francisco, 1893) should not have been forgotten. If three volumes of Father Hagen's *Synopsis der Höheren Mathematik* are to be listed with dates, the fourth, 1930, should also be given; when the authors refer as they did to Father Hagen's index to Euler's writings I wonder if they were not really thinking of Eneström's bibliography, so much better and complete. Reference to Cayley in a history of American mathematics is not inappropriate since he not only lectured in Baltimore for a few months, but also published twenty-five memoirs in the American Journal of Mathematics.

In the next section on *American dissertations*, are listed doctoral dissertations accepted before 1901 at Chicago, Columbia, Harvard, The Johns Hopkins, and Yale universities. The "general trend of mathematical research in America for the period 1875–1900" is mainly found by study of the Jahrbuch über die Fortschritte der Mathematik. In the last decade of the period, it appeared, for example, that in each of the fields, algebra and geometry, nearly twice as much was done as in each of the fields "mechanics and mathematical

* Other statements about Sylvester are either misleading or incorrect. On page 75 it is stated that at the University of Virginia Sylvester "failed as a teacher and left at the end of a year." As a matter of fact, Sylvester arrived at the University in November, 1841, and resigned on February 24, 1842; his resignation had nothing whatever to do with his teaching. On page 126 is an equally inaccurate statement, namely: "After a few months of continued trouble he was compelled to leave"; he was not "compelled" to leave. (Anyone wishing to get the correct facts in this regard may refer to P. A. Bruce, *History of the University of Virginia*, 1819–1919, New York, vol. 3, 1921, pp. 75–77.) The statement is repeated (later on p. 126) that as a teacher "he was a failure"; and yet a third time (p. 128), though not quite so strongly. Sylvester's wonderfully inspiring professorial activities at The Johns Hopkins University, and the following quotation from Bruce's *History*, seem to be adequate refutation: "He has a good deal of hesitation, is not fluent, but is very enthusiastic, and commands the attention and interest of his class." Reference is made to Sylvester's "*Collected Works*" (p. 128); this should be *Collected Mathematical Papers*. Sylvester's poem *Spring's Début, a Town Idyll* was dated (inside, p. 28) January, 1880; hence the statement (p. 126) "*n.d.* but *c.* 1880" is hardly accurate.

physics," and "history, biography, philosophy, teaching"; and more than twice as much as in the field of "theory of functions." "Trends of important branches" are chiefly exhibited for each of 14 topics by a chronological listing of published papers. With the final three-page "retrospect," and eight-page index, the work concludes.

Six full-page portraits of John Winthrop, Benjamin Peirce, Sylvester, E. H. Moore, Bôcher, and Gibbs add materially to the interest of the volume. There are many references to sources where further information may be found. It is hard to see how so much information could have been better condensed into two hundred small pages. On the whole the work is exceedingly valuable and suggestive, and American mathematicians must be highly grateful to the authors for thus notably contributing to their enlightenment and edification.

The following corrigenda submitted to me by Mr. S. A. Joffe, have been checked, and are put on record for a new edition:

p. 9, ll. 8–9, *for* "all the early American colleges" *read* "every American college";
p. 37, l. 14, *for* "of" *read* "on";
p. 38, l. 12, *for* "Mans" *read* "Man's";
p. 39, l. 9, *for* "shows" *read* "show";
p. 86, in the 11-line quotation there are too many inaccuracies to list;
p. 94, ll. 4–5 from bottom, *for* "automaton professors there begin" *read* "automata professors there, begin";
p. 99, ll. 1–2 of first footnote, *omit* "24" and *substitute* "670" *for* "570";
p. 106, l. 1 of footnote, *for* "become" *read* "became";
p. 109, *omit* ll. 4–5, and "pers:" in l. 6, and *insert* in l. 8, "Norbert Herz (Vienna)";
p. 192, l. 6 from bottom, *for* "Sur la logarithme" *read* "Sur le logarithme".

R. C. ARCHIBALD

SASULY ON STATISTICS

Trend Analysis of Statistics—Theory and Technique. By Max Sasuly. Washington, D.C., The Brookings Institution, 1934. xiii+421 pp.

The formal procedure for fitting polynomial curves by the method of least squares has been available, from the investigations of Legendre, Gauss, and Laplace, since the early part of the nineteenth century. However, its utility was, until lately, seriously limited by the excessive labor entailed in carrying through the numerical computations even for polynomial curves of relatively low degree. The most fruitful research in recent years has been directed to the development of an effective technique to simplify the numerical computation steps in cases in which the data are equally spaced and equally weighted. As a result of these investigations, the practical difficulties which were formerly encountered in these cases have been entirely removed.

The factors which have contributed mainly to the notable advance which has thus lately been achieved in facilitating the practical least squares fitting of polynomial curves are the use of the orthogonal form of polynomials in the fitting procedure, together with the employment of the Gregory-Newton expansions for these orthogonal polynomials. The advantages of the orthogonal representation had long been recognized and, in fact, the foundations of the theory of least squares orthogonal polynomial fitting had been laid by Tchebycheff in the middle of the last century. However, the practical application of his methods was very laborious. Then, in 1913, the practical least squares fitting of polynomial curves was greatly facilitated by the proposal of Sheppard to use the Gregory-Newton form of expansion for the equation of the polynomial curve to be fitted, that is, an expansion in a series of factorial terms, $1,\ x,\ x(x-1)/2!,\ x(x-1)(x-2)/3!,\ \cdots$, rather than in the usual form of a series of power terms, $1,\ x,\ x^2,\ x^3,\ \cdots$, owing to the fact that an explicit solution for the parameters of a polynomial of any degree could then be secured. Moreover, by using that form of expansion, the values of the parameters were obtained in terms of factorial moments of the data, which could readily be computed by a repeated summation procedure. The final stage in simplifying the practical least squares fitting of polynomial curves consisted in combining the advantages secured from the orthogonal representation with those resulting from the use of the factorial form of expansion of the polynomials.

In the book under review, the author has developed the theoretical basis of the underlying formulas for fitting polynomial curves by the method of least squares in much the same order as that which was actually followed in the course of the historical development, as briefly indicated above. That is, the formulas for least squares fitting of polynomial curves for a few of the lower degree polynomials in the familiar power series form are first derived. Up to the present time, however, no direct general solution for the values of the parameters of least squares power polynomials in terms of the power moments of the data appears to be available. Thus, attention is directed to the Gregory-Newton form of expansion, where a relatively simple, general solution in terms

of factorial moments of the data is obtained. The procedure for computing the factorial moments by means of repeated summations of the data is indicated. Finally, consideration is given to the least squares fitting of polynomial curves by the use of orthogonal systems of Gregory-Newton polynomials.

In addition to the development of methods for fitting polynomial curves to series of data by the method of least squares, the book also treats the smoothing of a series of data by means of moving polynomial arcs, and by functions of a somewhat broader character which may be derived from them.

In brief, then, it may be said that the purpose of this book is to exhibit the possibilities of trend representation by least squares polynomials, and of the smoothing, or graduation, of series of data by means of moving least squares polynomial arcs and closely related functions. The author states that the primary aim of this book is to "derive formulas and computation schedules that will simplify as much as possible effective, practical trend analysis . . . The theoretical basis of the underlying formulas is treated to an extent necessary to exhibit their derivation, and to disclose the simple relations between the new formulas introduced and others more familiar in the literature. This gives the necessary background as well as the full scope of the new method presented. However, the formulas and fitting procedure that are of primary interest to the computer may be applied independently of the formal proofs of their derivations . . . The procedure steps given in the illustrative examples . . . are given as nearly as possible in the order used in actual practice, and in sufficient detail so that their principal applications may be made without reference to the text discussion."

With this orientation concerning the general scope and aim of the book, it will perhaps be possible to proceed with clearer understanding to a detailed consideration of the contents.

The text is divided into three parts, or books. The title of Book I is *Fitted polynomials in trend analysis.* The first chapter in this book, Chapter 2, is devoted to general considerations concerning the fitting properties of polynomials, followed by a brief discussion of various types of fitting criteria, and closes with a general treatment of the problem of fitting functions by the criteria of least squares. The next chapter takes up the least squares fitting of mid-range power polynomials, and derives explicit formulas for the parameters in terms of the mid-range power moments of the data for polynomials through the sixth degree. A method for building up the orthogonal form of the mid-range power polynomials is indicated in the latter part of this chapter, and the explicit expression for the first ten of these polynomials is given. Chapter 4 deals with mid-range parameters and fitted values as weighted data averages. Here the points on the mid-range power polynomials are expressed explicitly in terms of the data to which the curve is fitted, and the values of the parameters are derived in terms of the data differences. The expressions for these higher order parameters can be used to determine certain important characteristics of the fitted curve, such, for example, as the slope and the curvature at various parts of the range. In Chapter 5, formulas are derived for interpolation by fitted mid-range power polynomials for any spacing interval. Chapter 6 discusses general polynomial interpolation by means of the Lagrange formula and the Newton divided difference formula, which are of use when the data

are unequally spaced. In Chapter 7, the last chapter in Book I, attention is directed to central difference interpolation by the Newton-Stirling, Newton-Bessel, and Everett formulas, and to smooth junction interpolation by means of the Karup cubic and Glover quintic formulas. The subjects of inverse interpolation and of approximate integration and summation are also treated in this chapter. In brief, then, Book I may be said to deal mainly with the least squares fitting of mid-range power polynomials, and with various methods of interpolation. For the most part, the results are in the form most familiar in the literature on these subjects.

The subject matter of Book II is *Trend determination by data sums and averages.* Chapter 8, the first chapter in Book II, deals with power moments and factorial moments of the data. The method for computing factorial moments by repeated forward summations of the data, as well as by repeated backward summations, is developed. The relations between factorial moments and power moments about the same origin, and also about arbitrary origins, are here derived. The chapter concludes with a discussion of moments and averages, in which formulas are derived for the values of certain useful statistical data averages in terms of factorial moments. Sheppard's moment corrections are also discussed here. Chapters 9 and 10 discuss the graduation of a series of data by means of moving least squares polynomial arc mid-values and iterated moving average functions. First, the procedure for computing the mid-values of moving least squares parabolic, quartic, and sextic arcs in terms of repeated summations of the data is indicated. However, the computation of moving arc mid-points becomes quite laborious for the higher degree arcs, and a method is indicated for building up various synthetic arc formulas which will secure approximately the same degree of smoothing as the moving sextic arc mid-value, but which is much easier to compute than the latter. These synthetic arc formulas involve the use of iterated moving averages, and a method is indicated for the computation of iterated moving average functions in terms of repeated summations of the data. Consideration is also given to the uses which can be made of graduation formulas in smoothing time series data. The remaining chapters in Book II deal with moment equivalence conditions for function arcs, and function parameters in terms of factorial moments.

Analytic trend fitting by general systems of polynomials is the subject matter of Book III. In the first chapter, Chapter 13, the useful Gregory-Newton form of expansion is employed to obtain the explicit expressions for the parameters of the least squares polynomial of any degree. Chapters 14 and 15 deal with orthogonal systems of Gregory-Newton polynomials. The method for building up such a system is indicated, the advantages of this form of representation in practical least squares polynomial curve fitting are set forth, and a convenient fitting procedure is developed. Chapter 16 takes up special polynomial trend fitting in integral form, and the last chapter in the book discusses interpolation and sub-tabulation by fitted polynomials. A group of tables for use in connection with the various procedures developed in the text are inserted at the end of the book.

This book should be of value both to the computer whose primary interest is in the acquisition of effective, practical, curve fitting methods, as well as

to those who may wish to follow the theoretical development of the effective methods which have lately been devised for fitting polynomial curves by the method of least squares. From the standpoint of this latter class, a very desirable feature of the book is the large number of references to source material.

It is to be regretted that no reference is made to three important papers on the least squares fitting of orthogonal polynomials which have recently appeared. It is possible, of course, that the latest of these papers had not been published at the time the manuscript of this book was completed. In the order of their appearance, these three papers are: (1) *On graduation according to the method of least squares by means of certain polynomials*, by Fredrik Esscher (Försäkringsaktiebolaget Skandia, 1855–1930, Stockholm, vol. 2 (1930), p. 107); (2) *Approximation and graduation according to the principle of least squares by orthogonal polynomials*, by Charles Jordan (Annals of Mathematical Statistics, vol. 3 (1932), p. 257); and (3) *On the graduation of data by the orthogonal polynomials of least squares*, by A. C. Aitken (Proceedings of the Royal Society of Edinburgh, vol. 53 (1933), p. 54). Esscher's paper is noteworthy primarily for its extremely simple development of the general form for the least squares orthogonal (Tchebycheff) polynomials. Esscher's development is very similar to that used by Goursat* in deriving the Legendre polynomials, to which the Tchebycheff polynomials are closely related. Jordan has published a number of papers on the subject of least squares orthogonal polynomial curve fitting, to some of which reference has been made in the book under review. His 1932 paper, which is mentioned above, may be said to round out his earlier work. In this paper he has appended some excellent bibliographical and historical notes which deal with the results obtained by other investigators in this field. Aitken's paper is outstanding not only for its simple and elegant development of the form of the Tchebycheff polynomials, which is similar to that given by Esscher, but also for presentation of extremely simple fitting procedure.

Throughout the book under review, the material is treated in a manner which makes few demands on the mathematical ability of the reader. In some instances, the method of development which is followed has resulted in the inclusion of perhaps an undue amount of detail. For example, in Chapter 8 it seems to us that the relationships between power moments and factorial moments, and between moments taken about different origins, could have been developed in a more compact and simple manner by making use of Vandermonde's theorem, together with the so-called differences of zero and differential coefficients of zero, tables of which are given later in Chapter 17, than by means of the somewhat laborious and extended algebraic procedure which was actually used. Likewise, the method which has been followed in resolving the least squares Gregory-Newton polynomials into orthogonal components in Chapters 14 and 15 is decidedly more involved and difficult, in our opinion, than the derivation of the orthogonal polynomials which is given in the above-mentioned papers of Esscher and Aitken.

The book contains a number of typographical errors. Fortunately, most of them are of such nature as to be readily apparent to the reader.

K. W. HALBERT

* Goursat-Hedrick, *Mathematical Analysis*, 1904, vol. 1, p. 173.

SHORTER NOTICES

L'Univers en Expansion. By Henri Mineur. (Actualités Scientifiques et Industrielles, No. 63.) Paris, Hermann, 1933. 41 pp.

This monograph is a good collection of the results obtained by the Lemaître-Eddington-de Sitter line of development of the theory of the "expanding universe." Like those whose work he reports, he is apparently unaware of the existence of Friedmann's 1924 article in the Zeitschrift für Physik on open spaces, and is unfamiliar with the reviewer's 1929 treatment of the subject in the Proceedings of the National Academy of Sciences.

H. P. ROBERTSON

La Théorie de la Relativité. By A. Einstein. Translated into French by Maurice Solovine. Paris, Hermann, 1933. 109 pp. +1 plate.

This pamphlet is a translation into French of three articles, two of which were published in German and the third of which was written by Einstein expressly for this collection. The first two, occupying the first 98 pages, are Einstein's fundamental 1916 article in the Annalen der Physik and the 1931 Einstein-Mayer Sitzungsberichte paper on the ("5-vector") unified field theory. The remaining article is an exposition of the general Friedmann universe (with the usual erroneous attribution of the open-space cases to Heckmann), with special emphasis on the case in which the cosmological constant and the curvature of space are both taken as zero.

H. P. ROBERTSON

Gleichgewichtfiguren Rotierenden Flüssigkeiten. By Leon Lichtenstein. Berlin, Springer, 1933. viii+175 pp.

This important work of the late Professor Lichtenstein was published not long before his death. In it he has given a somewhat brief historical account of work done on the problem by his predecessors, and a fuller account of the contributions made by himself and his students during the last few years. Since he presupposes on the part of his readers a familiarity with the theory of the newtonian potential function and of integral equations, together with a good grasp of other branches of analysis, geometry, and celestial mechanics, the book is not easy to read, but the results are so significant that it should be carefully studied by everyone who is seriously interested in the mathematical treatment of the problem of figures of equilibrium of rotating fluid bodies.

Suppose a body of fluid occupying a region T bounded by a surface S is rotating about an axis with an angular velocity ω. Under what circumstances will it be in relative equilibrium assuming that its parts attract each other according to the newtonian law? The importance of this problem from the point of view of the theoretical shape of heavenly bodies and of their evolution is obvious. It has been studied by a long line of famous mathematicians, including conspicuously Newton, Maclaurin, Jacobi, Clairaut, Tschebyscheff, Poincaré, G. H. Darwin, Liapounoff, Wavre, and the author of this book. If V is the newtonian potential of the body at a point P, and if R is the

distance of P from the axis of rotation, then for equilibrium the function $F=V+\omega^2R^2/2$ must be constant on the surface S. The potential V is expressed as a triple integral over the region bounded by S of a function of P and the variables of integration. Under such circumstances to determine S such that the resulting function F will be constant on S is clearly a problem of transcendental difficulty. Fortunately the function V is tractable for a homogeneous ellipsoid, and Maclaurin and Jacobi were able to show that if ω is small enough there are certain ellipsoids which are figures of equilibrium when the fluid is homogeneous. The relative lengths of the axes of these ellipsoids are functions of the angular velocity, there being two essentially different ellipsoids for each small angular velocity. As ω is varied these ellipsoids form two linear series of figures of equilibrium, the ellipsoids of the two series coinciding when ω has attained a certain size. This particular value of ω is thus a point of bifurcation, and is especially important since it suggests the possible existence of other linear series (not ellipsoids) which may have points of bifurcation with the ellipsoidal series. In 1884 Tschebyscheff, and in 1885 Poincaré (probably independently) made use of this idea to find new figures of equilibrium in the neighborhood of the ellipsoidal ones. Assuming the existence of solutions of a certain form, they proceeded to find approximations to the solutions, but in these celebrated papers the actual existence of the solutions was not established. About twenty years later Liapounoff, in a fine series of papers, completed the existence proofs and made very valuable extensions of the discussion of the general problem. His method depended in effect on the solution of a non-linear integral equation of a type discussed by E. Schmidt in 1908, but was made without the aid of the theory of integral equations.

Lichtenstein has made the solution of the problem depend on that of an integro-differential equation, and his method has the advantage of applying not only to solutions in the vicinity of the ellipsoidal solutions but also to those in the vicinity of any series of solutions. He is able to make extensive use of the theory of integral equations, which greatly simplifies the work as compared with that of Liapounoff. In the volume before us, he gives details of some of his existence proofs both for figures previously discussed and for certain new figures of equilibrium. His discussion includes not only the case of a homogeneous rotating fluid body, but also a considerable number of other interesting cases. For example, he discusses the case of a body made up of a finite number of layers each of which is homogeneous, the case of a rigid body covered with a homogeneous fluid, the case of a body with a variable density which varies monotonically from outside to the center, the case of figures which are approximations to anchor rings, and the case of two separate bodies. In this book he does not take up the question of stability of the figures of equilibrium, but references are given to memoirs on this very important matter. Nor does he discuss the exceedingly interesting question of the evolution of the form of a rotating fluid body whose density, mass, and rate of rotation change under conditions which might exist in the evolution of a celestial body; I regret that he did not find it possible to include at least historical references to this matter. A large number of problems closely related to those discussed in Lichtenstein's book await solution.

E. J. Moulton

Bessel Functions for Engineers. By N. W. McLachlan. Oxford, Clarendon Press, 1934. xi+192 pp.

This is the fourth volume of the Oxford Engineering Science Series. The purpose of the book is to provide engineers and engineering students with a course on Bessel functions and their practical applications. In conformity with this purpose the book is mainly concerned with expansions, formulas, and properties of the functions which are found useful in applications. The mathematical discussions are straightforward and formal with occasional use of the expression "it can be shown." The applications are mainly mechanical and electrical with particular attention to loud-speaker horns—a subject in which the author is particularly interested. The book contains about 600 examples (with answers) to be solved by the reader. At the end there is a rather extensive collection of formulas and brief tables of the principal functions.

H. B. PHILLIPS

Theorie den Konvexen Körper. By T. Bonnesen and W. Fenchel. Berlin, Springer, 1934. vii+164 pp.

This monograph is the first part of the third volume of the excellent series, Ergebnisse der Mathematik und ihrer Grenzgebiete, published by the Zentralblatt für Mathematik. It contains an exhaustive exposition of the theory of convex bodies in n-dimensional spaces, in all its ramifications and connections with differential geometry. Even a superficial perusal of the book will give the reader a clear idea of the importance and interest of this subject which unfortunately is not as widely known as it should be. A more attentive reading will reveal a multitude of results of a perfect beauty and will urge him to turn to a serious study of this fascinating field.

The monograph starts with exposition of fundamental notions of convex sets and bodies, of convex hulls, planes of support, centers of gravity, and classification of boundary points of a convex body (§§1–3). The relationship with the theory of convex functions and their applications to the problem of representation of convex bodies are shown next (§4). The following articles, §§5–6, treat of linear combinations of convex bodies, of linear and concave families of convex bodies, of convergent sequences of convex bodies, and of approximation of convex bodies by means of polyhedra and of analytic convex surfaces. Various important quantities and figures connected with convex bodies are discussed in §§7–8. Such are volume and mixed volume, cross sections, surface area, width, diameter, thickness, and the like. Special attention is given to the integral formulas for volumes and mixed volumes in terms of point coordinates and of functions of support (Stützfunktionen). Next, §§9-10 contain extensions of the methods of symmetrizing which were first introduced by Steiner in his classical investigations on the isoperimetric properties of a circle. Various extension problems and inequalities naturally belong here. Various proofs of the important theorem of Brunn-Minkowski and its numerous applications are treated in §§11-12. Next, §13 deals with the problem of determination of a convex body by means of its curvature functions, including uniqueness and existence theorems. Various special cases of convex bodies (such as convex bodies possessing a center, bodies of constant width, convex

quadric surfaces) are discussed in §§14, 15, and 16. The last article, §17, contains indications concerning the differential geometry in the large of convex curves and surfaces. The monograph closes with a rather complete bibliography. The reading of this remarkable monograph at places is not very easy, but is extremely suggestive and the total result is well worth the effort.

Being primarily interested in the theory of convex *bodies*, the authors did not enter into discussion of the theory of convex *functions* as such. The exposition of this theory, which plays such an important role in the modern development of analysis and theory of functions, would require an extended monograph of its own. Let us hope that such a monograph will appear soon in the Ergebnisse series, and that it will prove just as exciting as the monograph by Bonnesen and Fenchel.

J. D. TAMARKIN

Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche. By F. Enriques and O. Chisini. Volume 4, *Funzioni Ellittiche e Abeliane.* Zanichelli, Bologna, 1934. viii+274 pp.

This fourth volume concludes the series of magisterial lectures on algebraic geometry by Enriques with the collaboration of Chisini. They composed it during a vacation-sojourn in the country, and afterwards used it as the base of a course of lectures at the Universities of Rome and Milan, respectively.

The *Lezioni* are designed as preparatory for students who intend to take courses in which more advanced developments of algebraic geometry are given. In this task the authors have succeeded admirably. The student who masters Enriques and Chisini's lectures will have a foundation in algebraic geometry which cannot be matched anywhere else in the world.

Enriques himself promises to continue this work by treating in the same spirit algebraic surfaces or the algebraic functions of two variables.

The fourth volume contains the sixth book of the general treatise: *Elliptic and Abelian Functions.* In the first chapter, we find a very clear and concise treatment of elliptic integrals and functions with which every student of algebraic geometry should be familiar. One may of course expect that the geometric aspect of the theory is stressed in opportune places without impairing the rigor of the argument. Thus, the flex-configuration of the plane cubic, linear point-series and correspondences on elliptic curves are very adequately treated.

The second chapter is concerned with Abelian integrals and appears as an extension of the function-theoretical and geometric theories so beautifully explained in Chapter 1.

In the last chapter, the authors discuss the famous problem of inversion and Abelian functions. Here again we see at every turn the hand of master geometricians handling an otherwise purely analytical theory.

The volume ends with an application of Abelian functions to hyperelliptic and Kummer surfaces. As in the preceding volumes, at the end of every chapter we find a very competent and accurate account of the historical development of the theory.

Enriques and Chisini have written a beautiful book which may be strongly recommended to student and teacher alike.

ARNOLD EMCH

NOTES

The American Mathematical Society will hold its 1936 Summer Meeting on September 1–5, in connection with the Tercentenary Celebration of Harvard University. The Mathematical Association of America will meet on August 31. Harvard University is inviting distinguished mathematicians from each of four countries, and is generously scheduling the lectures of these guests to take place before the Society and the Association. Professor E. W. Chittenden will deliver the Colloquium Lectures on Topics in General Analysis. There will be a joint meeting with the American Astronomical Society. Arrangements will be made to house a considerable number of the visiting mathematicians in the Harvard dormitories, and there will be numerous excursions to points of historic interest in Cambridge and the neighborhood. This meeting should prove one of the most interesting and important that the Society has held.

The Mathematical Institute of the University of Moscow has organized a topological conference to take place September 4–10. American mathematicians who expect to attend and take an active part are Professors J. W. Alexander, Solomon Lefschetz, John von Neumann, P. A. Smith, M. H. Stone, Hassler Whitney, D. V. Widder, and Oscar Zariski, Doctors E. R. van Kampen and H. W. Tucker, and Mr. Garrett Birkhoff.

Representatives of the American Mathematical Society at the Seventh American Scientific Congress, sponsored by the Department of Public Education of the Republic of Mexico, September 8–17, will be Professors M. A. Basoco, of the University of Nebraska, and E. V. Huntington, of Harvard University. Professor E. L. Dodd, of the University of Texas, has been appointed as a delegate of the Mathematical Association of America, and will present a paper entitled *A survey of statistical means or averages.*

Professor Oystein Ore, of Yale University, has been appointed a member of the editorial board of the Duke Mathematical Journal.

The Section on Mathematics (A) of the American Association for the Advancement of Science met on June 25 at the University of Minnesota in conjunction with the Section on Physics (B). Professors W. L. Hart, W. E. Brooke, and J. T. Tate presided successively. About sixty members and guests were present. The program consisted of three invited addresses, on the general theme of properties of differential equations important in quantum mechanics. The titles of the papers and their authors were as follows: *Oscillation theorems associated with boundary value problems*, by Professor R. W. Brink, of the University of Minnesota; *The Stokes phenomenon*, by Professor R. E. Langer, of the University of Wisconsin; *Applications of approximation methods to physical problems, with particular reference to molecular spectra*, by Professor D. M. Dennison, of the University of Michigan. A joint luncheon of Sections A, B, and D in the Minnesota Union Tuesday noon was attended by forty persons.

The company was addressed briefly by President Compton of the Association, who was introduced by the Chairman of Section B, Professor J. T. Tate.

Dr. R. C. Tolman, professor of mathematical physics at the California Institute of Technology, was elected president of the Pacific Division of the American Association for the Advancement of Science at the Los Angeles meeting.

The International Mathematical Congress, under the presidency of Professors Alf Guldberg and Carl Størmer, will be held at Oslo July 13–18, 1936. This date early in the summer has been set in order to avoid conflict with the Summer Meeting of the American Mathematical Society mentioned above. The Congress will be organized in the following sections: (1) Algebra and Theory of Numbers; (2) Analysis; (3) Geometry and Topology; (4) Theory of Probability, Insurance, Statistics; (5) Astronomy; (6) Mechanics and Mathematical Physics; and (7) Philosophy, History, and Education. Nearly a score of leading mathematicians have been invited to give general lectures before the Congress, including the following representatives from American universities: Professors G. D. Birkhoff, Oystein Ore, Oswald Veblen, and Norbert Wiener. Titles of papers intended for presentation at the Congress should be sent to the Sécrétaire Général, Edgar B. Schieldrop, Universitetet, Oslo, Norvège. After the Congress, trips will be arranged to some of the picturesque spots in Norway.

The section of physical sciences of the Royal Academy of Bologna announces the following subjects for its two Adolfo Merlani prizes in mathematics: 1. *Contributions to the study of roots of the integrals of linear differential equations (in the real domain)*; 2. *A contribution relating to any question dealing with the mechanics of fluids, and preferably of aerodynamics.* Competition is restricted to members of the academy and closes December 31, 1936.

At its annual commencement, Harvard University conferred the honorary degree of doctor of science on Professor Albert Einstein of the Institute for Advanced Study at Princeton.

Dr. Irving Langmuir, of the General Electric Company, has recently been elected a foreign member of the Royal Society of London.

Professor Solomon Lefschetz, of Princeton University and president of this Society, has been elected a member of the Royal Society of Letters and Sciences of Bohemia.

The honorary degree of doctor of science was conferred on Professor Marston Morse, of Harvard University, by Colby College.

Professor Oystein Ore, of Yale University, delivered lectures on structure theorems and problems in abstract algebra at Zürich, Göttingen, and Hamburg, as well as in the Academy of Sciences at Oslo during his trip abroad.

Professor P. F. Rider, of Washington University, has been awarded a General Education Board Fellowship. He will spend a year carrying on research

in biomathematics and mathematical statistics, principally in the Galton Laboratory at University College, London.

Professor D. J. Struik, of the Massachusetts Institute of Technology, delivered lectures on the theory of probability and the imbedding of a Riemann manifold in a euclidean space of higher dimensions at the Mathematical Institute of the University of Moscow, U.S.S.R. Other guests of the Mathematical Institute were Professors J. A. Schouten of Delft and T. Levi-Civita of Rome.

Dr. R. S. Zug, assistant professor of mathematics and astronomy at Drake University, has been awarded a grant from the National Research Council in order to work on galactic star clusters.

A Committee of colleagues, friends, and former students of Professor J. Hadamard has been organized to celebrate his scientific jubilee which will probably take place next December. The Secretary and Treasurer of the Committee are respectively Maurice Fréchet and G. Valiron. It is planned to publish a volume of 250 to 300 pages including selected reprints of Professor Hadamard and also a pamphlet containing the names of subscribers and the addresses to be given on the occasion of the celebration. Moreover, the 1937 volume of the Journal de Mathématiques will be dedicated to him.

A similar Committee, the president of which is Émile Picard, has been organized to celebrate the scientific jubilee of Professor Édouard Goursat. A medal will be presented to him at a ceremony to which subscribers will be invited. Subscribers will receive a bronze replica of the medal. The 1936 volume of the Journal de Mathématiques will be dedicated to him.

The former pupils of Marcel Brillouin are planning to present him with a jubilee volume in honor of his eightieth birthday. The volume will combine the original scientific memoirs of many of his old pupils. Subscribers will be entitled to a copy of this book.

The following have been awarded National Research Fellowships in Mathematics for 1935–36: J. A. Clarkson, Norman Levinson, W. T. Martin, F. J. Murray, S. B. Myers, J. B. Rosser, G. C. Webber. This list includes renewals.

Professor H. S. Carslaw, of the University of Sydney, has been named professor emeritus of mathematics.

Dr. Wilhelm Cauer, of the University of Göttingen, has been promoted to a professorship of mathematics.

Dr. E. T. Copson, of Royal Naval College, Greenwich, has been appointed to a chair in mathematics in University College, Dundee.

Dr. R. H. Fowler, Plummer professor of mathematical physics at Trinity College, Cambridge, England, has been appointed visiting lecturer in mathematics at Princeton University for the second term of the next academic year.

Dr. Theodor Vahlen has been appointed professor of mathematics at the University of Berlin.

Dr. H. L. Dorwart, of Williams College, has been appointed assistant professor of mathematics at Washington and Jefferson College.

Professor R. L. Jeffery, of Acadia University, will be visiting professor of mathematics at the University of Wisconsin next year.

Dr. C. B. Morrey, of the University of California, Berkeley, has been promoted to an assistant professorship in mathematics.

Professor W. F. Osgood, formerly of Harvard University, will remain at the National University of Peking for another year.

Dr. Gordon Pall, of McGill University, has been promoted to an assistant professorship.

Dr. A. E. Pitcher, of Harvard University, has been appointed an assistant at the Institute for Advanced Study.

Associate Professor G. Y. Rainich, of the University of Michigan, has been promoted to a professorship.

Professor A. N. Shaw has been made head of the department of physics and director of the MacDonald Physics Laboratory at McGill University. He succeeds Professor A. S. Eve who retired at the close of the academic year.

Associate Professor G. W. Smith, of the University of Kansas, has been appointed to a professorship.

Professor J. H. Van Vleck, of the University of Wisconsin, has been appointed to a professorship of mathematical physics at Harvard University.

The following appointments to instructorships are announced: University of Kentucky: Dr. E. D. Jenkins; Mississippi State College: Mr. W. E. Cox, Jr.; Yale University: Mr. Marshall Hall.

Professor J. G. Estes, of North Carolina State College, died on June 1, 1935.

The death of Dr. Alfred Loewy, formerly professor of mathematics at the University of Freiberg, is reported.

Rev. J. de Séguier, of Paris, died on April 19, 1935. He had been a member of this Society since 1902.

Professor B. M. Wilson, professor of mathematics in University College, Dundee, formerly lecturer in pure mathematics in the University, died on March 18, 1935, at the age of thirty-eight years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

292. Dr. F. G. Dressel: *A note on a linear operational equation.*

The paper presents solutions of the equation $L(u)+f(x)P(u)=0$, where $L(u)$ is a functional operator with properties similar to those of a homogeneous linear differential operator, and $P(u)$ is a linear operator. When the methods of this paper are applied to the problem considered by R. P. Gillespie (Proceedings of the Edinburgh Mathematical Society, (2), vol. 4 (1935), pp. 80–84) more complete results are obtained than those presented in Gillespie's paper. (Received July 24, 1935.)

293. Dr. J. S. Frame: *The simple group of order* 25920. Preliminary report.

This paper extends the results of a paper *On the irreducible representations of hyperorthogonal groups* (presented to the Society December 28, 1934, and soon to appear in the Duke Mathematical Journal). A certain infinite family of simple groups, $HO(m, q^2)$, may be represented by unitary matrices of degree m, with coefficients from a finite field $GF(q^2)$ of q^2 elements, where q is the power p^s of a prime p, and conjugates are defined by $\bar{x}=x^q$. The vectors a with components a_i from the $GF(q^2)$, for which $\sum a_i a_i=0$, are permuted among themselves and so are those for which $\sum a_i a_i=k\neq 0$. A set of monomial representations, which were found to be irreducible when $m=3$ are no longer so when $m>3$, but always split into 2 or 3 irreducible components. Only one of the simple groups in question is of order less than three million. This is the group $HO(4, 4)$, of order 25920. It can be generated by two operators of orders 2 and 5 respectively, whose product is of order 9. It is analyzed into its 20 sets of conjugate operations, the characters of 6 reducible monomial representations are split into irreducible components, and finally the complete table of characters for the group is obtained. (Received July 29, 1935.)

294. Mr. C. B. Tompkins: *An integral invariant of n-dimensional shells in* $(2n-1)$*-space.*

A method is presented for determining an integral invariant associated with a homeomorph of an n-dimensional sphere lying in euclidean $(2n-1)$-

space. An integer is obtained by means of an integration over a contour, an $(n-1)$-dimensional simple closed variety which does not bound on the shell. For a special set of standard shells the invariant is shown to represent the number of twists of the shell, the number of times any pair of non-intersecting contours interlink. The particular problem is to start with the shell defined locally by means of a generalization of the Gauss formulas, and to obtain from the coefficients of these formulas an integral invariant which is determined by the shell. The invariant was found by Rainich when $n=2$. The method depends upon finding the solid angle described by the vector lying in the shell perpendicular to a contour, this vector referred to invariant directions of the contour, while its base point passes over the contour. The invariant directions are the mean curvature vector and vectors obtained by repeated application of the generalized Laplacian, depending on coefficients of the Gauss formulas as connection coefficients, on this mean curvature vector. (Received August 1, 1935.)

295. Dr. C. W. Vickery: *Concerning distance functions and point functions.*

In the present treatment is considered a generalization of the notion of distance satisfied by various connected spaces, including certain spaces in which there exist uncountable convergent sequences of points. Suppose that with each pair of points (A, B) is associated a type μ sequence $\delta_\mu(AB)$ of positive real numbers (i.e., a point of space E_μ) satisfying certain conditions analogous to metric conditions of Fréchet; such a space is called a space Δ_μ'. It is shown that spaces E_μ, Δ_μ, and $U_{\eta_1\omega_1}$ are spaces Δ_μ', Δ_μ', and Δ_{ω_1}' respectively. Necessary and sufficient conditions for spaces Δ_μ' in terms of point and region are considered. Also functions are considered which map a point set P on a point set Q in spaces Δ_{ω^α}'. Oscillation is defined such that a function is continuous at p_0 if and only if its oscillation at p_0 is ϕ $(0, 0, 0, \cdots)$. The set of points of discontinuity is a set $F_\sigma\aleph_\alpha$ relative to P (sum of $\aleph_\alpha$ sets closed relative to P) and the set of points of continuity is a $G_\delta\aleph_\alpha$ relative to P (product of $\aleph_\alpha$ open subsets of P). (Received August 2, 1935.)

296. Dr. C. W. Vickery: *Spaces of uncountably many dimensions.*

In this treatment are considered spaces D_ν, E_ν, and E_ν', the set of all points of which is the set of all type ν sequences of real numbers, ν being an ordinal number. The definitions of sequential limit point are the same for spaces D_ν and E_ν as for spaces D_ω and E_ω, respectively, of Fréchet. It is shown that for every ν, D_ν and E_ν are arc-wise and locally arc-wise connected. If ν is an element of $Z(\aleph_\alpha)$, then D_ν is homeomorphic with D_{ω_α} and E_ν is homeomorphic with E_{ω_α}. For every ν, D_ν is metric. Space D_{ω_α} is not $\aleph_\chi$-separable for any $\aleph_\chi < 2^{\aleph_\alpha}$. For $\nu \geqq \omega_1$, E_ν is not metric. Region is defined in space E_{ω_α}, and limit point in terms of region, in such a way that the original meaning of sequential limit point is preserved; this extends the notion of limit point so that there are uncountable convergent sequences of points (for $\nu \geqq \omega_1$); the resulting space is called E_{ω_α}'. Space E_{ω_α}' is a distributive space $\Gamma(\aleph_\alpha)$ (see author's paper, Tôhoku Mathematical, Journal vol. 40 (I) (1935), pp. 1–26). (Received August 2, 1935.)

297. Professor A. T. Craig: *Note on a certain bilinear form that occurs in statistics.*

In this paper, we investigate some of the properties of the distribution of a real symmetric bilinear form of $2n$ variables normally and independently distributed. (Received August 4, 1935.)

298. Professor C. C. Craig: *A new exposition and chart for the Pearson system of frequency curves.*

By making the quantities α_3 and δ (*The Handbook of Mathematical Statistics*, H. L. Rietz, Editor-in-Chief, p. 104) fundamental in the discussion, a new exposition of the Pearson system of frequency functions is developed which the author believes possesses marked advantages in unity, clarity, elegance, and convenience. The chart corresponding to the Rhind-Pearson diagram is strikingly simple and clear. (Received August 5, 1935.)

299. Professor C. C. Craig: *Sheppard's corrections for a discrete variate.*

By the use of moment generating functions and the argument used by R. A. Fisher in the case of a continuous variable, the corrections to the moments derived by J. R. Abernethy (Annals of Mathematical Statistics, vol. 4 (1933), pp. 263–277) for grouping in the case of a single discrete variable are derived in a considerably more elegant manner. The method is immediately adapted to two and more variables and the results likely to be useful are given for two variables. The method gives first the corrections to the semi-invariants (of Thiele) which are much simpler in form. The known corrections for continuous variables are obvious limiting forms of the results given. (Received August 5, 1935.)

300. Dr. Ben Dushnik: *Some theorems on the maximum sum of ordinals.*

Consider a sum of the type $\Sigma \equiv \Sigma_{\beta<\alpha} a_\beta$, where α is a specified ordinal number and, for every ordinal $\beta<\alpha$, a_β is likewise a specified ordinal number. The ordinal α is called the argument of the sum Σ, while the set of different ordinals which appear as summands in Σ, each with its "multiplicity," may be called the "system of summands" of the sum Σ. Two systems of summands are identical only if they comprise the same set $\mathfrak{A}$ of ordinals, and if each ordinal of $\mathfrak{A}$ has the same multiplicity in both systems. When one specifies the argument α and the system of summands, one may still obtain different sums Σ corresponding to different arrangements. Thus, the two sums $\Sigma_1=\omega^4+\omega^3+\omega+\omega+\cdots+\omega$ and $\Sigma_2=\omega^3+\omega+\omega+\cdots+\omega^4$, where ω is the least transfinite ordinal, have the same argument, $\omega+1$, and the same system of summands: ω^4, ω^3, ω, with multiplicities 1, 1, $\aleph_0$, respectively; yet $\omega^4+\omega^3+\omega^2+\omega=\Sigma_1>\Sigma_2=\omega^4$. The present note considers whether, for a given argument and system of summands, there exists a "maximum" sum Σ, and if so, what arrangement of summands gives this maximum (in the above example, Σ_1 is the greatest sum). (Received August 5, 1935.)

301. Professor M. H. Ingraham: *Characteristic spaces associated with the matrix whose elements belong to a division algebra.*

If M is a matrix with coefficients in a field F, then the rank of M and of $N=T^{-1}MT$ are equal, and for every polynomial g the rank of $g(M)$ is equal to the rank of $g(N)$ and the space of vectors ξ such that $g(M)\xi=0$ has the same rank as the space of vectors η such that $g(N)\eta=0$. If, however, M, T, N, and ξ have elements belonging to a division algebra or to a quasi-field D, and if $g(x)=\sum x^i d_i$ where the d's are in D, then the above theorem no longer holds. A process of multiplication of such polynomials and an operation with such polynomials, and associated matrices upon vectors is defined so as to re-establish much of the theory connected with matrices and their characteristic values, characteristic vectors and invariant factors and associated spaces. This theory leads to great simplifications of the theory for the similarity of matrices whose elements belong to a division algebra. (Received August 5, 1935.)

302. Dr. W. T. Martin (National Research Fellow): *Geometrical representation of the growth of entire functions of several complex variables.*

We say that the function $F(z_1, z_2, \cdots, z_n)=\sum_{m_1,\cdots,m_n=0}^{\infty} a_{m_1\cdots m_n} z_1^{m_1}\cdots z_n^{m_n}$ is of *exponential type* if the associated series defined by the formula $f(z_1, \cdots, z_n)=\sum_{m_1,\cdots,m_n=0}^{\infty} a_{m_1\cdots m_n}[(m_1+\cdots+m_n)!]z_1^{m_1}\cdots z_n^{m_n}$ possesses a ($2n$-dimensional) region of convergence. We obtain for such functions F analogues of certain of the results obtained by G. Pólya (Mathematische Zeitschrift, vol. 29 (1929), pp. 549–640) for functions of a single variable. With every function F of exponential type we associate a real-valued function $h(\alpha_1, \cdots, \alpha_n)$, ($\alpha_1, \cdots, \alpha_n$ arbitrary complex numbers), which measures the growth of the function F in the direction of the half ray which joins the origin to the point $(\alpha_1, \cdots, \alpha_n)$. By means of the function h we associate with the function F a uniquely determined open region D in the space R_{2n}. The region D contains the invariant convergence region of the power series development of the function f and is an "invariant" region (invariant under linear transformations of coordinates!). Every analytic (2-dimensional) plane P through the origin cuts D in a region in which f is regular and which is the "best" region in P (of a given character) in which f is regular. (Received August 5, 1935.)

303. Professor A. D. Michal: *Notes on Riemannian and non-Riemannian differential geometry in abstract spaces.*

Sets of postulates for Riemannian and non-Riemannian differential geometry in abstract spaces were first given by the author in 1933. The consequences of these postulates were developed sufficiently far to yield a theory of parallel displacement and curvature. In the present paper new postulates are added to some modified sets of postulates and we study such subjects as abstract Riemannian geometry with torsion, abstract Riemannian and non-Riemannian geometries with distant parallelism, and abstract geometries with a *class* of generalized Ricci tensors. It is worth while to mention here that there exist

infinitely dimensional functional geometries with a *continuous infinitude* of generalized Ricci functional tensors. (Received August 5, 1935.)

304. Dr. J. F. Randolph: *On generalizations of length and area.*

Definitions of p-dimensional measure of q-dimensional sets have been given by C. Carathéodory, W. Gross, and others. In this paper is discussed, without answering it completely, the simplest phase of the fundamental question whether the generalizations of length and area under these definitions preserve, as do Lebesgue's, the euclidean relation that area is the product of length by length. Let B be a plane point set and A the set of all points (x, y, z) such that (x, y) is in B and z lies between zero and a constant h. If B is C-linearly measurable we show that B is also G-linearly measurable (the converse is not true), that the space set A is both C- and G-plane measurable, and furthermore that the C-plane measure of A is less than or equal to h times the C-linear measure of B and the inequality is reversed for G-measures. Consequently for the restricted class of sets B for which the C- and G-linear measures are the same, the "area" of A is the "base" times the altitude. Finally we obtain a condition for the equality of the C- and G-linear measures of B, but show that this condition is not necessary. (Received August 5, 1935.)

305. Miss V. E. Spencer: *A classification of certain sequences of polynomials associated with persymmetric determinants.*

A sequence of polynomials $\{\Phi_i(x)\}$ is associated with any sequence of persymmetric determinants $\{\Delta_i\}$ by replacing the last row of Δ_i by $1, x, x^2, \cdots, x^{i-1}$. Let the numerical value of $\{\Delta_i\}$ be $\{\delta_i\}$. A set S is defined as a set of sequences $\{\Phi_i(x)\}$ with each of which is associated the same $\{\delta_i\}$. If a set S contains $\{\Phi_i(x)\}$ it contains also $\{\Phi_i(x-m)\}$, m arbitrary. Any set S contains one and only one symmetric sequence. A study is made of the matrices transforming one sequence of S into another. Invariants of S are found. Orthogonal Tchebycheff polynomials (O.T.P.) form a subclass of all sets S. In connection with the theory new bounds for the zeros of O.T.P. are exhibited. (Received August 5, 1935.)

306. Mr. John W. Wrench, Jr.: *Sequences of transformations in a general metric space.*

This paper is concerned with the general problem: If two sequences of point sets (M_i) and (N_i) having sequential limiting sets M and N respectively are given in a metric space, and if, for each i, T_i is a transformation such that $T_i(M_i) = N_i$, then under what conditions on T_i, M_i, and N_i does there exist a transformation T such that $T(M) = N$? Under sufficiently strong uniformity conditions, the transformation T is a homeomorphism; under weaker conditions T is (1–1) and continuous, but its inverse may not be continuous; under still weaker conditions it can merely be stated that T is (1–1). The paper is an extension of one by H. M. Gehman (Proceedings of the National Academy of Sciences, vol. 18 (1932), pp. 460–465). (Received August 5, 1935.)

307. Professor M. M. Culver: *Fundamental regions in S_{12} for the simple collineation group of order* 1092 *on seven variables.*

A certain set of fourteen Hermitian forms are selected which are permuted by the operators of this group. The ninety-one differences of these forms, equated to zero, are regarded as hypersurfaces in S_{12}. A point which lies on exactly seventy-eight of these hypersurfaces is selected. In a sufficiently small neighborhood of this point there are exactly 13! regions. From this fact it can be shown that the whole set of hypersurfaces divides the S_{12} into 14! regions. These 14! regions may be regarded as consisting of 1092 sets of $2 \cdot 11!$ each. Thus there are 1092 fundamental regions in S_{12} for the group in question. Of seven complex variables there exist sets of fourteen linear forms whose absolute values, for suitably chosen values of the variables, may be given any desired arrangement in order of magnitude. (Received August 6, 1935.)

308. Dr. J. H. Curtiss: *Interpolation in non-regularly distributed points.*

In a paper to appear in the Transactions of this Society (see abstract 40-9-263) the author studied the convergence of the sequence of polynomials which coincide with an integrable function in the sequence of "regularly distributed" sets of points on a Jordan curve C. This present note discusses the use of other possible choices of the points of interpolation in this problem, and in particular, describes in some detail the results obtained when the points are transforms of the roots of unity under maps other than the map which transforms the roots of unity into regularly distributed points on the curve C. (Received August 6, 1935.)

309. Mr. I. E. Highberg, Professor A. D. Michal, and Mr. A. E. Taylor: *Abstract euclidean spaces.* III.

In this paper the authors continue their study of a certain abstract space E with independently postulated "geometric" and "analytical" metrics (see abstracts 40-11-383 and 41-3-158). A set of sixteen independent postulates is given. The co-ordinate methods of Hilbert space are not applicable and several results, trivial in Hilbert space, take on an important and difficult aspect. The following theorem, among others, is proved on rotations: Let $y=x+T(x)$ be a proper rotation, and let $\Gamma(x)$ be the transformation associated with the inverse, $x=y-\Gamma(y)/2$, of $y=x+T(x)/2$; then $\Gamma(x)$ is "skew-symmetric." Conversely, if $\Gamma(x)$ is a "skew-symmetric" transformation, such that $x=y-\Gamma(y)/2$ has the unique solution $y=x+T(x)/2$ for each x in E, then $x+T(x)$ is a proper rotation. Some of the key theorems and methods of Hilbert space as developed by Hilbert, Riesz, Stone, von Neumann, and others in connection with projection operators, characteristic value problems, and resolution of the identity do not go over to abstract euclidean spaces. Several of the theorems that go over to E are proved by new methods. (Received August 6, 1935.)

310. Professor A. D. Michal and Mr. D. H. Hyers: *An existence theorem for a second order differential system with two-point boundary conditions in general analysis.*

In this paper an existence theorem for the differential equations $d^2x(t)/dt^2 = F(t, x, dx/dt)$ in complete normed vector spaces is proved. An application

is then made to the theory of geodesics in Riemannian and non-Riemannian geometries in abstract spaces. Some attention is also given to function space instances of the general theory. (Received August 6, 1935.)

311. Professor D. W. Woodard: *Sufficient conditions that a space be an n-dimensional manifold.*

This paper is based upon results obtained by the author in a preceding paper (see abstract 40-7-236). For definitions of the terms and symbols employed in the theorem which follows, the reader is referred to the above-mentioned paper which has been submitted to the Transactions for publication. The present work, which makes use of a result obtained by G. Nöbeling (Monatshefte für Mathematik and Physik, vol. 42, p. 145), is concerned with the proof of the following theorem: A θ-space M^n is an n-dimensional manifold provided the following conditions are fulfilled: (1) There exists at least one neighborhood $N_{\alpha\beta}$ such that, if there is a homeomorphism, $H_1[\lambda(N_{\alpha\beta})]=\lambda(N_{\alpha\beta})$, then there exists a homeomorphism, $H_2(\overline{N}_{\alpha\beta})=\overline{N}_{\alpha\beta}$, of such nature that $H_2[\lambda(N_{\alpha\beta})]=H_1[\lambda(N_{\alpha\beta})]$. (2) Let W_1 and W_2 be two sets such that $W_1\cdot W_2\neq 0$, $W_i\cdot \not\supset W_j, i\neq j$, W_1 is a neighborhood $N_{\phi\psi}$, W_2 is an element of the set $\Delta^a_{i-1}[P_i]N_{\gamma\delta}$, where $0\subseteq P_i\subseteq\sum_{k=1}^{b}\lambda(N_{\alpha_k\beta_k})$, $\alpha_k\leqq\gamma$, and, if $W_2\neq N_{\gamma\delta}$, $\phi\leqq\gamma$. Then, $\overline{W}_2\cdot\lambda(W_1)=\sum_{j=1}^{t}\overline{C}_j^{n-1}$, $\Delta^t_{j=1}[C^{n-1}_{\beta_j}]W_2\neq 0$ and every component of $\lambda(W_2)-\lambda(C_i^{n-1})$ is an $(n-1)$-cell. (Received August 6, 1935.)

312. Professor H. A. Davis and Dr. Amos Black: *Some non-involutorial space transformations associated with pencils of nodal cubic surfaces.*

Transformations defined by means of a one-to-one correspondence between the surfaces of two pencils of nodal cubic surfaces and the points of certain rational curves are discussed. Their association with certain transformations of Pieri and Caldarera is shown. (Received July 31, 1935.)

313. Dr. Amos Black: *A series of involutorial Cremona space transformations defined by a pencil of ruled cubic surfaces.*

A series of transformations is defined by means of a correspondence between the surfaces of a pencil of ruled cubic surfaces and the points of certain rational curves, called the director curves. The director curve is not part of the basis of the pencil; one of the principal surfaces is a ruled surface R, all of whose generators are parasitic lines; all the surfaces of the homoloidal web have fixed tangent planes along a certain curve, but none of the fixed planes is determined by R. (Received July 8, 1935.)

314. Professor Hassler Whitney: *Differentiable manifolds in euclidean space.*

If M is an m-dimensional manifold defined by overlapping neighborhoods, and if the relation between the two given sets of coordinates in the common part of two neighborhoods is one with continuous derivatives through the rth order ($r\geqq 1$) and with non-vanishing Jacobian, we say M is of class C^r.

We show that M may be mapped in a 1–1 differentiable manner (in fact in a "C^r-manner") in euclidean space E^{2m+1}. Any such M in any E^n may be approximated to by an analytic manifold in E^n, the approximation being closer and closer as we approach the boundary (if there is one). Hence M may be given an analytic Riemannian metric. An outline of the paper will be found in the Proceedings of the National Academy of Sciences for July. (Received July 8, 1935.)

315. Professor C. G. Latimer: *Note on the quadratic subfields of a generalized quaternion algebra.*

Let $\mathfrak{A}$ be a rational generalized quaternion algebra with the fundamental number d. Since every element of $\mathfrak{A}$ satisfies a quadratic equation with rational coefficients, $\mathfrak{A}$ contains an infinitude of quadratic fields. In this note it is shown that necessary and sufficient conditions for $\mathfrak{A}$ to contain a given quadratic field F are: (a) F is imaginary if $d>0$, and (b) no rational prime factor of d is the product of two distinct prime ideals in F. Fueter showed that (b) is a necessary condition (*Quaternionenringe*, Commentarii Mathematici Helvetici, vol. 6 (1934), p. 199). (Received July 11, 1935.)

316. Dr. W. C. Randels: *On Volterra-Stieltjes integral equations.*

An existence theorem is proved for the integral equation $f(x)=g(x)+\int_0^x f(y)\,d_yK(x,y)$. The principal restriction on $K(x,y)$ is that there exists a monotone function $V(y)$ such that $|K(x,y_2)-K(x,y_1)|\leqq|V(y_2)-V(y_1)|$. An example is given to show that this restriction cannot be replaced by $\int_0^1|d_yK(x,y)|<1$. (Received July 11, 1935.)

317. Dr. W. C. Randels: *On a theorem of Plessner.*

Plessner has shown that if $f(x)\subset L_2$ and $\{a_n\}$, $\{b_n\}$ are the Fourier coefficients of $f(x)$, then $\sum_{n=2}^{\infty}(a_n\cos nx+b_n\sin nx)/\sqrt{\log n}$ converges almost everywhere. If we denote by $E(Pl,f)$ the set where Plessner's series converges, then we show that $E(Pl,f)\not\subset E(L,f)$, $E(L,f)\subset E(Pl,f)$ where $E(L,f)$ is the Lebesgue set of $f(x)$. (Received July 11, 1935.)

318. Professor G. Szegö: *Inequalities for the zeros of Legendre polynomials and related functions.*

Denoting the zeros of the Legendre polynomial $P_n(\cos\theta)$ in $(0,\pi)$ (in increasing order) by $\theta_1,\theta_2,\cdots,\theta_n$, Bruns, and later in a sharper form, A. Markoff and Stieltjes (independently) have given various estimates for θ_ν. In the first part of the present paper Sturm's classic method is used to obtain the inequalities $\pi(\nu-1/4)/(n+1/2)<\theta_\nu<\nu\pi/(n+1)$, $\nu\leqq g=[n/2]$. The upper bound is the same as that of Markoff-Stieltjes, the lower one is even better. As a consequence of Sturm's method, the convexity of the sequence $0=\theta_0,\theta_1,\theta_2,\cdots,\theta_{g+1}$ is stated, this fact being used in the proof of the inequalities mentioned. Sturm's method leads also to various other estimates of the zeros of P_n as well as of related functions. The second part of the paper deals with a

general distribution theorem for the zeros of the trigonometric polynomials $\sum_{k=0}^{m} \lambda_k \cos(m-k)t$ with $\lambda_0 > \lambda_1 > \cdots > \lambda_m > 0$. According to Pólya these are all real and simple. Now, each interval $\pi(v-1/2)/(m+1/2)$ $\pi(v+1/2)/(m+1/2)$, $(v=1, 2, \cdots, m)$, contains exactly one of these zeros. This gives an estimate for θ_v also, although not as good an estimate as Sturm's method. It can, however, be improved by direct treatment. (Received July 11, 1935.)

319. Professor Alonzo Church and Dr. S. C. Kleene: *Formal definitions in the theory of ordinal numbers.*

The theory of formal definition (λ-definition) of functions of positive integers, developed by Kleene (American Journal of Mathematics, vol. 57 (1935), pp. 153–173, 219–244), is extended to the transfinite ordinals. Formulas are assigned to represent the ordinals in the first number class and those in a certain subset of the second number class, plausibly identified as the set of "constructive" ordinals in the second number class. To each ordinal in this subset of the second number class, however, an infinite number of non-interconvertible formulas is assigned, and there is no means by which it can be effectively determined, in general, whether two given formulas represent the same ordinal. The notion of formal definition of functions of ordinals is introduced in the manner of Kleene, p. 219, and the formal definability of certain familiar functions of ordinals is established. It is pointed out that, in the case of any formally definable function of ordinals, the process of reduction of a formula to normal form provides an algorithm for calculating the values of the function (in the sense of calculating formulas which represent those values). In conclusion a method is outlined of extending the theory to greater ordinals. (Received July 13, 1935.)

320. Professor T. R. Hollcroft: *Properties of branch-point manifolds associated with linear systems of primals.*

A linear r-parameter system of primals (hypersurfaces) F of order n in S_r is associated with an involution whose properties and those of F are determined by the characteristics of the branch-point primal L of the involution. The following theorem is proved: The characteristics of the tangent cone from S_k to L, the branch-point primal associated with a linear r-parameter system of primals in S_r, are respectively the characteristics of the branch-point manifold L_{r-k-2} associated with a linear $(r-k-1)$-parameter system of primals contained in the linear r-parameter system. The values in terms of n of the order and all $r-1$ classes of L are obtained and the classes of all sections of L by linear manifolds. (Received July 15, 1935.)

321. Professor H. L. Rietz: *On the frequency distributions of certain ratios.*

Certain properties of the frequency distributions of the ratios y_i/x_i, when x_i and y_i are drawn from normal universes, have been the subject of study of several authors, notably Greenwood, Karl Pearson, C. C. Craig, and Greary. The present paper continues the study by finding the frequency functions of $t=y_i/x_i$ where x_i and y_i are taken at random from very simple distributions

limited in several cases to a finite part of the xy-plane, and noting some of their properties. (Received July 18, 1935.)

322. Professor Gordon Pall: *Binary quadratic discriminants differing by square factors.*

The Lipschitz correspondence, by means of linear transformations of determinant p, between binary quadratic forms of discriminants d and p^2d is combined with precise relations among the numbers of sets of representations of n and p^2n in a class, genus, or order. Incidentally the number of ambiguous classes derivable from a given ambiguous class is determined. (Received July 19, 1935.)

323. Professor F. Riesz and Dr. E. R. Lorch: *The integral representation of unbounded self-adjoint transformations in Hilbert space.*

The paper deals with unbounded self-adjoint transformations in Hilbert space. It embodies two short proofs of the formula $A=\int_{-\infty}^{\infty}\lambda dE(\lambda)$ for such a transformation. Previous proofs have all been based on the theory of bounded transformations. But none of these makes systematic use of the theory of such transformations, specifically to the extent of making use of the above formula and its immediate consequences for bounded transformations. The point of view of the paper is to derive the formula, exploiting as fully as possible results concerning bounded transformations well known for over twenty years. The first proof demonstrates the existence of $E(\lambda)$, the so-called resolution of the identity of A, but does not allow of a simple construction for it. The second proof gives a construction of $E(\lambda)$ in terms of elementary operations on A. (Received July 23, 1935.)

324. Dr. F. G. Dressel: *Solutions of bounded variation of the Volterra-Stieltjes integral equation.*

The purpose of this paper is to give conditions under which the Young-Stieltjes integral equation $f(x)=g(x)+\lambda\int_0^x K(x,y)\,df(y)$ has a solution $f(x)$ of bounded variation. (Received July 24, 1935.)

325. Mr. D. M. Dribin: *Representation of binary forms by sets of ternary forms.*

A. A. Albert has introduced the concept of the representation of integers by sets of forms and has studied the representation of positive integers by the set $\sum(d)$ of all positive classic ternaries of determinant d. O. K. Sagen, in his dissertation at Chicago, has treated a similar problem for non-classic ternaries. In the present paper, the representation of a given classic binary, $\phi=(a, t, b)$, $t^2-ab\neq 0$, by a given set $\sum(d)$ of all classic ternaries of determinant d is studied. By a theorem of Gauss, the problem of representation of a given binary by $\sum(d)$ can be reduced to one of *proper* representation. Hence attention need be confined only to this latter case and in the present paper the problem of proper representation is completely solved. Of the results obtained the following theorem may be cited: A primitive binary ϕ of determinant

$\delta \neq 0$ is represented by a set $\sum(d)$ of all classic ternaries of determinant d if and only if $(-d\phi|p)=1$ for every odd prime p dividing the kernel of δ and prime to d. (Received July 24, 1935.)

326. Professors A. H. Copeland and Francis Regan: *A postulational treatment of the Poisson law.*

A set of postulates relating to probability distributions for time series are presented in this paper. These postulates determine the character of a function $f(\alpha, E)$ where $f(\alpha, E)$ is the probability that there will be α points of the series within a given Lebesgue measurable set E. It is proved that, for a special case, this function has the classical form of the Poisson law. (Received July 26, 1935.)

327. Professors Virgil Snyder and Evelyn Carroll-Rusk: *The Veneroni transformation in S_n.*

The purpose of this paper is to obtain the set of bilinear equations in terms of which the transformation may be expressed, to obtain the complete set of multiple loci on the image of a general prime, and to map the lines, surfaces, and other varieties on straight lines, planes, three way spaces, and so on. A number of these properties have been obtained previously for S_4 but by means that could not be applied to the general case. (Received July 26, 1935.)

328. Dr. C. C. Torrance: *Paratingents in topological spaces.*

Point sets, called lines, having properties analogous to euclidean lines are defined in a non-metric topological space S by means of an extension of the concept of upper semi-continuous collection. Paratingents to point sets in S are defined by means of these lines after the manner of Bouligand. The theorem is proved that the paratingent to a point continuum C at a point of C is itself a line continuum. (Received July 26, 1935.)

329. Professor H. M. Gehman: *On extending a homeomorphism between two subsets of spheres.*

This paper is concerned with proving a theorem on the extension of a homeomorphism between two subsets of spheres similar to a theorem previously proved for subsets of planes (Transactions of this Society, vol. 31 (1929), pp. 241–252). The relationship between extension theorems for plane sets and for subsets of spheres is discussed. (Received July 29, 1935.)

330. Professor H. A. Davis and Dr. Amos Black: *A non-involutorial Cremona transformation belonging to the complex of secants of a twisted cubic.*

A transformation is defined by two pencils of quadric surfaces and a regulus of bisecants of the twisted cubic, all put into projective relation. The same transformation may be defined by pencils of cubic surfaces or nets of quartic surfaces. (Received July 31, 1935.)

331. Dr. L. A. Wolf: *Similarity of matrices in which the elements are real quaternions.*

This paper gives a necessary and sufficient condition that two matrices, A and B, of which the elements are real quaternions, be similar; that is, that there exist a non-singular matrix S whose elements are real quaternions such that $SAS^{-1}=B$. This paper defines a set of invariant factors for any matrix A, of which the elements are real quaternions, in terms of the ranks of certain real polynomials in A. The condition for similarity of two such matrices, A and B, is that the invariant factors of $(A-\lambda I)$ and $(B-\lambda I)$ be the same. (Received July 31, 1935.)

332. Professor Dunham Jackson: *Variations on Bernstein's theorem.*

In the theorem of Bernstein on the derivative of a polynomial in a finite interval the ends of the interval present a singularity which does not appear in the corresponding theorem on the derivative of a trigonometric sum for unrestricted real values of the variable. The author has shown (Transactions of this Society, vol. 26 (1924), pp. 139–145) that a proposition involving an analogous singularity holds for a trigonometric sum considered over a part of a period. The present paper gives a simplified proof of this result, and applies the same method to the derivation of a sequence of other theorems of similar character. (Received August 1, 1935.)

333. Professor A. J. Kempner: *Waring's problem and diophantine equations.*

In the author's thesis *Über das Waringsche Problem und einige Verallgemeinerungen* (Göttingen, 1912) it is noted that every problem of Waring type is completely equivalent to a certain linear diophantine equation with restrictions on the solutions. For example, "every positive integer = a sum of four or fewer squares," is equivalent to "the equation $n=1\cdot x_1+3\cdot x_2+5\cdot x_3+\cdots$ always has positive integral solutions with $4\geqq x_1\geqq x_2\geqq x_3\geqq\cdots$." On account of the growing importance of inequalities in many fields, and in view of the large number of Waring problems which have been solved, it seemed to the author worthwhile to collect the corresponding information available concerning diophantine equations. This is done in the present paper. (Received August 1, 1935.)

334. Professor A. J. Kempner: *Remarks on "unsolvability."*

This paper represents an effort of the author to make clear the mathematical and psychological difficulties involved in "unsolvability" problems. Two points have particularly attracted his attention. (a) It seems to the author that it is not frequently enough pointed out that there are essentially different ways in which a problem may be "unsolvable," and that this makes it unnecessarily hard for those not specializing in "Grundlagenforschung" to follow the literature. (b) Problems suggested by Brouwer and others suffer from the psychological handicap that one sees no mathematical reason why

they should possibly be "unsolvable." The author attempts to outline the domain which, in discussions on unsolvability, even the strict classicist must concede to the intuitionist as unsettled. (Received August 1, 1935.)

335. Professor A. J. Kempner: *Anormal systems of numeration.*

The introduction of number systems to a base g not a positive integer leads to some interesting results and problems concerning uniqueness of representation, periodicity of decimal fractions, etc. These are considered in the present paper. (Received August 1, 1935.)

336. Professor Leonard Carlitz: *Note on higher congruences.*

In this paper are collected some results on congruences of the type $A_0t^{p^{nk}}+A_1t^{p^{n(k-1)}}+\cdots+A_kt\equiv B \pmod{P}$, where A, B, P are polynomials in an indeterminate x with coefficients in a Galois field of order p^n. The simplest case is that in which the associated operator $A_0E^k+\cdots+A_k$, where $Ey=y^{p^n}$ for all y, is factorable (mod P) into linear factors. Certain results are also obtained concerning the number of possible factorizations. (Received August 3, 1935.)

337. Professor W. G. Warnock: *Note on line configurations.*

In a previous paper (Tôhoku Mathematical Journal, vol. 36 (II) (1933), pp. 303–319) the author considered line configurations derived from the identity $p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23}=0$ under group transformations. In this note attention is given to configurations formed by the intersections of these lines with the fundamental planes of the tetrahedron of reference of space S_3. In particular, a necessary and sufficient condition is given for the intersections to lie by pairs on rays through the vertices of the fundamental triangle of each face of the tetrahedron. (Received August 3, 1935.)

338. Mr. Aaron Herschfeld: *The equation* $2^x-3^y=d$.

L. E. Dickson's *History of the Theory of Numbers* records that Leo Hebreus, or Levi Ben Gerson (1288–1344), proved that all powers of 2 and 3 differ by more than unity except the pairs 1 and 2, 2 and 3, 3 and 4, 8 and 9. This result was extended in 1918 by G. Pólya who showed that there can be only a finite number of solutions of the equation $a^x-b^y=d$, where a and b are any fixed positive integers and d is a given integer not equal to zero. In 1931 S. S. Pillai derived an asymptotic formula for the number of solutions of the inequality $0<a^x-b^y\leq n$. In the present paper the author finds all the solutions (x, y) of the inequality $|2^x-3^y|\leq 10$ by solving the twenty-one equations $2^x-3^y=d$, $-10\leq d\leq 10$. In addition it is proved that if $|d|$ is sufficiently great the equation $2^x-3^y=d$ can have at most one solution. (Received July 12, 1935.)

339. Dr. C. B. Tompkins: *A note on continuous deformations locally one-to-one.*

In order to complete the results of an earlier paper, the author seeks a transformation which will untie knots in, but will not untwist, a homeomorph

of an n-dimensional spherical shell in $(2n-1)$-space. Such a transformation is one which is the result of a continuous deformation locally one-to-one, defined in the following way. A transformation T transforming the variety X to Y is the result of a continuous deformation locally (1, 1) if there exists a set of transformations $S(\lambda)$ satisfying the conditions: (1) $S(\lambda)$ is a transformation defined uniquely for $0 \leqq \lambda \leqq 1$; (2) $S(0) = I$, the identity transformation; (3) $S(1) = T$; (4) the transform of each point of X as a function of λ is continuous; and (5) X may be covered by a finite number of neighborhoods of such a nature that on each neighborhood $S(\lambda)$ is a homeomorphism for all values of λ. Under such transformations all $(n-1)$-dimensional spheres with continuous derivatives are equivalent and every two topological shells with the same number of twists are equivalent. (Received August 2, 1935.)

340. Professor Leopold Fejér: *Trigonometric series and power series with repeatedly monotonic coefficients.*

Using previous results concerning the positive character of finite trigonometric sums or of the partial sums of higher order of some trigonometric series and the line of argument of G. Szegö, the author derives various facts concerning the positive and monotonic character of trigonometric series with terms in $\cos n\theta$, $\sin n\theta$, $\cos (2n-1)\theta$, or $\sin (2n-1)\theta$. Here the coefficients are real null sequences satisfying a monotonicity condition of a definite order. Under the same condition the univalence of power series is proved and estimates of the partial sums are given. For example: If $\{c_n\}$ is monotonic of order 4, the function $\sum c_n \cos n\theta$ is monotonically decreasing in $(0, \pi)$. Under the same condition $\sum c_n z^n$ is univalent in $|z| < 1$. Further investigation has been devoted to the "remainders," $\sum_{\tau=0}^{\infty} c_\tau \sin (n+2\nu+1)\theta$. If $\{c_\nu\}$ is monotonic of order 3, this function has at least n zeros in $(0, \pi)$. Each of the intervals $(\nu-\frac{1}{2})\pi/n$, $\nu\pi/(n+1)$, $\nu \leqq [(n+1)/2]$, contains at least one of the zeros. This result gives a new and elementary proof of the estimates of A. Markoff and Stieltjes for the zeros of Legendre polynomials $P_n(\cos\theta)$. (Cf. a previous paper of G. Szegö.) In the present proof Heine's development of P_n in sines is used, and a new proof of this development is given. (Received August 16, 1935.)

341. Professor Dunham Jackson: *Some problems of closest approximations by means of trigonometric sums.*

In a recent abstract the writer has referred to some extensions of Bernstein's theorem, in which a characteristic feature of the demonstrations is the use of overlapping intervals covering the range of the independent variable, in connection with a transition from the case of polynomials to that of trigonometric sums. In the present paper the same device is used in proving theorems for normalized trigonometric sums and for trigonometric approximation analogous to those given for polynomials in the writer's paper on *Certain problems of closest approximation* (this Bulletin, vol. 39 (1933), pp. 889–906), together with some new results for the polynomial case. (Received August 22, 1935.)

342. Professor Harold Hotelling: *Relations between two sets of variates.*

Between two sets of normally correlated variates $x_1, \cdots, x_s, y_1, \cdots, y_t$

the correlations invariant under internal linear transformations are roots $\rho_1, \cdots, \rho_s$ (where $s \leqq t$) of a certain determinantal equation. The number of non-vanishing roots is the number of independent common components of the two sets. The primary statistical problem here is equivalent to the algebraic problem of finding invariants and standard forms for two quadratic forms in the x's and y's respectively and a bilinear form in the x's and y's, under transformations of the two sets separately. The solution of the algebraic problem gives the canonical forms $\sum_1^s x_i^2, \sum_1^t y_i^2, \sum_1^s \rho_i x_i y_i$. For statistical purposes it is also necessary to determine sampling distributions of these roots and of functions of them. Some of these sampling problems are solved completely in this paper, and approximations and partial solutions are given for others. In order to throw light on these sampling problems, contributions are made to the geometry of the curved four-dimensional manifold in 1–1 correspondence with the lines in 3-space, and of generalizations of this manifold. Certain problems of statistical inference involving a plurality of unknown parameters are discussed. Applications are numerous, including the problems of common components in mathematical psychology, anthropology, economics, meteorology, and other subjects. (Received August 7, 1935.)

343. Professor Gordon Pall: *On rational automorphs of binary quadratic forms.*

The denominator of any rational automorph of $f=[a, b, c]$ which carries an integral solution of $ax^2+bxy+cy^2=n$ into an integral solution must be a divisor of n. Any two integral solutions of this equation can be transformed into each other by proper (and improper) rational automorphs of f. (Received August 12, 1935.)

344. Dr. C. L. Siegel: *Mean values of arithmetic functions in number fields.*

Let $f(\xi)$ be a function defined for all integers of a totally real algebraic field $\mathfrak{K}$ of degree n. To ξ let correspond in the n-dimensional space the point whose coordinates are the n conjugate numbers to ξ. For any domain G in that space we form the sum $F=\sum_{\xi \text{ in } G} f(\xi)$, in which ξ assumes all integral values in $\mathfrak{K}$ for which the associated point ξ is situated in G. In some investigations of number theory an asymptotic expression of F, if G becomes in certain ways infinitely large, is sought. The content of this paper is a formula which generalizes a well known formula for the sum of the coefficients of a Dirichlet series. An application to the problem of divisors is given. (Received August 14, 1935.)

345. Dr. C. L. Siegel: *The volume of the fundamental domain for some infinite groups.*

The purpose of the paper is the proof of a formula which gives an explicit expression for the volume of fundamental domains for certain classes of infinite groups. A special application gives the non-euclidean volume of the fundamental domain for the modular group in every totally real algebraic field. Blumenthal and Hecke have shown the importance of the corresponding modular functions for algebraic arithmetic investigations. Since the knowledge of a

set of generators of the modular group is necessary for the construction of any example in the theory of modular functions, the determination of the volume of the fundamental domain can be useful for further researches. (Received August 14, 1935.)

346. Dr. C. L. Siegel: *On the algebraic integrals of the restricted three-body problem.*

Among the few general recent results in the problem of three bodies the theorem of Bruns deserves special interest in spite of its negative character. The statement is that the ten well known integrals constitute the complete field of algebraic integrals of the problem. The present paper proves an analogous result for the restricted problem of three bodies. The result cannot be obtained by specializing Bruns' theorem. (Received August 14, 1935.)

347. Mr. Benjamin Rosenbaum: *Generalizations of some theorems of Schur on irreducible polynomials.*

In this paper it is shown that polynomials $f(x)=1+\sum_{\nu=1}^{n-1} g_\nu a!x^{l\nu}/(k\nu-b)! \pm a!x^{ln}/(kn-b)!$, as well as polynomials $g(x)$ which are given by the formula $g(x)=1+\sum_{\nu=1}^{n-1} g_\nu u_{2a}x^{l\nu}/u_{2(k\nu-b)} \pm u_{2a}x^{ln}/u_{2(kn-b)}$, where $u_{2j}=1\cdot 3\cdot 5\cdots(2j-1)$ and where g_ν, a, b, k, l, n are rational integers with certain restrictions, are irreducible in the rational field. The restrictions are: $1\leqq a\leqq(kn-b)/2$; $0\leqq b\leqq kn-2$; $k, l\geqq 1$; $n>1$. When $k\nu-b<1$, replace it by 1. This is a generalization of the results of I. Schur (Berliner Sitzungsberichte (1929), pp. 125, 370). It is further shown that $f(x)$ and $g(x)$ are special cases of an irreducible polynomial defined by the divisibility properties of the coefficients. For certain values of $kn-b$ the irreducibility of $f(x)$ and $g(x)$ can be established by means of the Eisenstein and Koenigsberger criteria. (Received August 22, 1935.)

348. Dr. L. M. Blumenthal (National Research Fellow): *Isometric geometry methods in determinant theory.*

Determinants of the type $|r_{ij}|$, $r_{ij}=r_{ji}>0$, $(i\neq j)$, $r_{ii}=0$, $r_{0i}=1$, $(i\neq 0)$, $(i, j=0, 1, \cdots, m)$, and of type $|r_{ij}|$, $r_{ij}=r_{ji}$, $r_{ii}=1$, $(i, j=1, 2, \cdots, m)$, are studied in this paper, and new theorems concerning them are presented. These theorems are obtained by applying results in the metric characterization of n-dimensional euclidean, spherical, and hyperbolic spaces. (Received August 24, 1935.)

349. Mr. I. E. Perlin: *Sufficient conditions for a minimum in the problem of Lagrange with isoperimetric conditions.*

In this paper sufficient conditions are established for a minimum in the parametric problem of Lagrange with isoperimetric side conditions. These conditions can be deduced directly from those for an associated non-parametric Lagrange problem without isoperimetric side conditions. These results are established by the aid of a generalized Lindeberg theorem which is developed in this paper. Two generalized Osgood theorems applicable to the

parametric problem of Lagrange with isoperimetric side conditions are developed. The methods and results employed in this paper are those of Tonelli and McShane. No use whatsoever is made of the notion of a field. (Received September 3, 1935.)

350. Dr. C. W. Vickery: *Concerning spaces of uncountably many dimensions.*

The author has shown that every $\aleph_\alpha$-separable metric space is homeomorphic with a subset of space D_{ω_α} and that every $\aleph_\alpha$-completely separable, normal, topological space is homeomorphic with a subset of space E_{ω_α}. Thus in order that a space be metric it is necessary and sufficient that it be homeomorphic with a subset of some space D_{ω_α}. Spaces D_{ω_α} and E_{ω_α} were defined by the author in a paper presented to the Society recently. (Received September 3, 1935.)

351. Professor N. H. McCoy: *On the characteristic roots of matric polynomials.*

Let $A_1, A_2, \cdots, A_m$ be finite matrices of order n, and P the algebra of polynomials in these matrices with coefficients in the field of complex numbers. The following three properties are shown to be equivalent: (1) the characteristic roots of every polynomial $f(A_1, A_2, \cdots, A_m)$ are all of the form $f(\lambda_1, \lambda_2, \cdots, \lambda_m)$, where λ_i is a characteristic root of A_i $(i=1, 2, \cdots, m)$; (2) every matrix $A_iA_j-A_jA_i$ $(i\neq j)$ is zero or properly nilpotent in P; (3) there exists a non-singular matrix R such that each matrix RA_iR^{-1} is in triangle form. Special cases of one or more of these properties have been studied by Bruton, Ingraham, Roth (this Bulletin, abstracts 38-9-196, 38-9-197, 41-3-168, respectively), and Williamson (American Journal of Mathematics, vol. 57 (1935), pp. 281–293). Direct generalizations of the results mentioned are also obtained for the case in which it is required that just $r(\leqq n)$ of the characteristic roots of every polynomial be of the form stated in (1). The proofs are based on theorems of E. Noether on the representations of algebras. (Received September 5, 1935.)

352. Dr. Max Zorn: *Discontinuous groups in topological spaces.* Preliminary report.

For groups G, discontinuous (in the sense of Baer-Levi) in a connected, locally connected, separable, locally compact space T, there exists a fundamental domain F with the following important property: If a set S of points in the closure $\overline{F}$ of F, plus all its images under G, has a limit point, then S has already a limit point. If $\overline{F}$ is compact, G has a finite number of generators. If T is a locally euclidean space, T and $\overline{F}$ may be considered as complexes; as a special result we obtain the characterization (Sperner, Kerékjártó) of plane translations. The following application to hypercomplex arithmetics is also discussed: the units of an order in a semi-simple algebra over the rational numbers form a group, which has a finite number of generating elements with a finite number of defining relations. (Received September 5, 1935.)

353. Dr. Max Zorn: *On Schur's generalization of Fermat's theorem.*

Schur defined the derivative of the sequence $a_0, a_1, \cdots, a_n$ with respect to the rational prime p as the sequence of quotients $a_{n+1}-a_n/p^{n+1}$. The derivative of the sequence $1, a, a^p, \cdots$, where $a_{n+1}=a_n^p$, is integral (Fermat). Schur proved the first $p-1$ derivatives to be integral for $(a, p)=1$, and got some further results. In this paper the meaning of Schur's theorems in p-adic analysis is given. A simpler proof is obtained in terms of p-adic logarithms and exponential functions. This proof gives in addition the congruence value mod p of numbers which are integers after Schur. (Received September 5, 1935.)

354. Professor J. W. Lasley: *On Monge's differential equation.*

This paper presents a solution of a differential equation of Monge by means of the theory of linear dependence. Since the differential equation in question has an historical setting of uncommon interest, the salient facts are noted. (Received September 5, 1935.)

355. Mr. Garrett Birkhoff: *A new definition of limit.*

Let H be any Hausdorff space. An aggregate Σ of sets of points of H will be said to *converge* to a point p of H if and only if (1) the set-theoretical product of any two sets of Σ contains a suitable set of Σ, and (2) any neighborhood of p contains a suitable set of Σ. By modifying this definition one can (1) construct a topological group out of the homeomorphisms of any Hausdorff space with itself; (2) complete any topological group or topological linear space; and (3) complete any *multiply distanced space.* (Received September 5, 1935.)

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WEISSENBERG (E.). See HOLZER (W.).

WOLF (K. L.). See FUCHS (O.).

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, October 26, 1935.

Abstracts must be in the hands of Associate Secretary J. R. Kline, Low Memorial Library, Columbia University, New York City, not later than September 28. Abstracts received by the Secretaries by September 6 will appear in the September issue of this BULLETIN. By invitation of the Program Committee, Professor J. L. Walsh will deliver an address on *Interpolation by rational functions.*

LEXINGTON, KENTUCKY, November 29-30, 1935.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 1. All these will appear in the November issue of this BULLETIN. By invitation of the Program Committee, addresses will be delivered by Professor L. L. Dines on *Convex domains and linear inequalities,* and by Professor C. G. Latimer on *The arithmetic of generalized quaternions.*

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, November 30, 1935.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than November 2. All these will appear in the November issue of this BULLETIN.

ST. LOUIS, MISSOURI, ANNUAL MEETING, December 31, 1935-January 3, 1936.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 26, 1935. Abstracts received by the secretaries by November 2 will appear in the November issue of this BULLETIN. This meeting will be held in conjunction with meetings of the A.A.A.S., and the Mathematical Association of America. Professor Vannevar Bush, of the Massachusetts Institute of Technology, will deliver the twelfth Josiah Willard Gibbs Lecture. By invitation of the Program Committee, addresses will be delivered by Professor Szegö on *Some recent investigations concerning the sections of trigonometric and related series,* and by Professor Synge on *Tensorial methods in dynamics.* On Tuesday afternoon, at a joint session with the Mathematical Association and Section A of the A.A.A.S., Professor Carmichael will deliver his retiring address as Chairman of Section A, on *Linear differential equations of infinite order.*

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or Low Memorial Library, Columbia University, New York City.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1934, p. 66), are payable to the Treasurer of the Society, Professor G. W. Mullins, Low Memorial Library, Columbia University, New York City.

WHOLE NUMBER 432

CONTENTS

PAGE

Ernest Brown Skinner—In Memoriam. By E. B. Van Vleck 593

Wiener on the Fourier Integral. By Einar Hille 595

Quine on Logistic. By Alonzo Church 598

American Mathematics before 1900. By R. C. Archibald 603

Sasuly on Statistics. By K. W. Halbert 607

Shorter Notices 611

Notes 615

Abstracts of Papers 619

New Publications 637

For official Communications and Notices, see the inside of the back cover.

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